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CYCLE TIME ESTIMATION FOR SIMULATING A TANDEM QUEUEING SYSTEM USING AGGREGATION TECHNIQUES

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ABSTRACT

One approach to simulating a single-server tandem queuing system is to explicitly model each of the production stages. In this paper, we apply queueing theory, a recursive algorithm, and composite random number sampling to develop an equivalent aggregate representation consisting of only a single production stage. Preliminary test results indicate that the aggregation works well for estimating the mean and variability of the total cycle time.

1 MODEL DESCRIPTION

We consider a tandem queueing system consisting of a series arrangement of a finite number of production stages or resources. The machine component of each resource has one server and each server can operate on one part at a time and has internal storage for that part. The parameters of this tandem queueing or flow line system can be summarized as follows:

$$FL = \langle R, R_1, ..., R_N, S \rangle$$
$$R = \langle 1/\lambda, Z \rangle$$
$$S = \langle U \rangle$$
$$R_i = \langle Q_i, M_i \rangle \quad i = 1, ..., N$$
$$Q_i = \langle v_i, x_i \rangle \quad i = 1, ..., N$$
$$M_i = \langle f_i, m_i, s_i, c_i \rangle \quad i = 1, ..., N$$

A flow line (FL) consists of three primary components, a receiving area (R), a shipping area (S), and N production stages or resources (R_i). This relationship is illustrate in Figure 1. The receiving area (R) is described by the mean time between arrivals ($1/\lambda$), where λ is the arrival rate, and Z, which is the maximum number of parts that can arrive from the storage area. The shipping area (S) is characterized by its storage capacity (U). Assume parts arrive according to a Poisson process and that Z and U are infinite.

<<<< **Figure 1 Approximately Here** >>>>

Each resource (R_i) consists of a queue (Q_i) and a machine (M_i) which is to service (*i.e.,* process, inspect, or machine) a part. The queue component is the waiting space proceeding the process where a part waits on a first-in-first-out basis until the single-server ($c_i = 1$) becomes available to process it. Assume the buffer capacity (x_i) is infinite and that v_i is the variability of the time between part arrivals to the queue. The time to service a single part for each of the machines is

specified by a probability distribution (f_i) and its corresponding mean (m_i) and standard deviation (s_i).

Friedman (1965) proposes a reduction procedure for tandem queues based on the dominance of a queue's impact on the other queues of the flow line. Applying his procedure results in modeling only the dominant queues of the system. The other, less dominant queues are represented by only their service means. In comparison, we propose that in a reduced representation of a single-server tandem queuing system, all resources are aggregated together to form a single aggregate resource, AR_1. Figure 2 illustrates the resulting aggregation flow line.

<<<< **Figure 2 Approximately Here** >>>>

An aggregate flow line consists of the receiving area (R), the shipping area (S), and one aggregation resource (AR_1). The queue (Q_1^*) of the aggregation resource is assumed to have infinite storage capacity. The machine, M_1^*, represents all the single-server machines of the original system. The machine is characterized by its service time distribution (f_1^*) and its corresponding service mean (δ_1^*). Note that f_1^* represents all the aggregated service time distributions. Developing a process for estimating this aggregate service time distribution is the objective of our analysis.

One approach for determining the aggregate resource service time distribution is to develop a combined or joint probability distribution using the original service time distributions (f_i). Unfortunately, since general (*i.e.,* non-exponential) service time distributions are allowed, a combination may be infeasible, inefficient, or impossible to develop. Our solution is to represent this unspecified service time distribution not as a single mathematical function, but rather as a relationship between the original service time distributions using a procedure known as the

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composition or *mixture method* (Law and Kelton 1991). Kronmal and Peterson (1979) explain that some continuous distributions are efficiently generated by representing them as mixtures of several other (continuous) distributions that are easy to generate. As such, we propose to never specify the aggregate resource service time distribution (f_i^*), but rather, to sample values from it during the execution of the aggregate simulation model. The advantage of our approach over Friedman's is that the variability for each of the individual service time distributions remains represented. This is significant in that the service time variability is often one of the key characteristics of a system (Pegden et al. 1990).

Pritsker (1986) summarizes that composite sampling assumes that "the density function must be written as a weighted sum of component distribution functions with the sum of the weights totaling one." That is, to sample from the unknown aggregate resource service time distribution (f_1^*) requires determining a weighting relationship between the original service time distributions (f_i). To find the distribution weights requires a three-step process. The first step, explained in Section 2, estimates the total waiting time represented by an aggregation resource. The second step, discussed in Section 3, computes the average service mean for the aggregate resource. The final step, explained in Section 4, determines the relative strength or weight of each of the original service time means towards the aggregate resource service mean. Special issues involved in specifying the aggregate simulation model are presented in Section 5. Preliminary results from aggregating a series of test scenarios are presented in Section 6. Section 7 provides a brief summary and discusses a limitation of the current methodology.

2 ESTIMATING CYCLE TIME

The first step in determining the distribution weights is to estimate the cycle time of the original flow line system. Since our analysis assumes that parts arrive to the first resource, R_1, by a Poisson process, the cycle time for a part at R_1 can be estimated by the Pollaczek-Khinchine formula for an M/G/1 queue (Kleinrock 1976):

$$E[T_1] = \frac{\lambda\left(m_1^2 + s_1^2\right)}{2\left(1 - \rho_1\right)} + m_1$$

where: $E[T_1]$ Expected cycle time for the first resource (R_1)
  λ Arrival rate to R_1
  m_1 Average service time of R_1
  s_1^2 Service time variation for R_1
  ρ_1 Traffic intensity at resource R_1: $\rho_1 = \lambda m_i$

Burke (1956) showed that the output of an M/M/S queue is Poisson. If the service time distribution of R_1 is exponential, its output process (arrival process to R_2) will also be Poisson with the same parameter values. The cycle time for subsequent resources in the flow line can be computed using the M/G/1 formula until the cycle time is computed for a non-exponential resource. Subsequent resource cycle time estimates would use a G/G/1 (general arrival and general service) queueing formula. Kumura (1991) proposes the following approximation:

$$E[T_i] = \left[\frac{cv_{a_i}^2 + cv_{m_i}^2}{2} g\left\{\frac{\rho_i}{1/m_i - \lambda}\right\}\right] + m_i \quad i = 2,...,N$$

where: $E[T_1]$ Expected cycle time of resource R_i (i = 2,...,N)
  λ Mean arrival rate
  $cv_{a_i}^2$ Coefficient of variation of the time between arrivals for R_i
  μ_i Average service rate for resource R_i (i = 2,...,N)
  $cv_{m_i}^2$ Coefficient of variation of the service time for resource R_i (i = 2,...,N)

ρ_i Traffic intensity at resource R_i: $\rho_i = \dfrac{\lambda}{\mu_i}$ (i = 1,...,N)

g

$$g\left(\rho_i, cv_{a_i}^2, cv_{m_i}^2\right) = \begin{cases} Exp\left[-\dfrac{2(1-\rho_i)}{3\rho_i}\dfrac{(1-cv_{a_i}^2)^2}{cv_{a_i}^2+cv_{m_i}^2}\right], & cv_{a_i}^2 \le 1 \\ 1, & cv_{a_i}^2 \ge 1 \end{cases} \quad i = 2,...,N$$

To estimate the cycle time using the above formula requires knowing the squared coefficient of variation of the arrival process. To determine this variation, it is necessary to explore the output process of a single-server queue. Marshall (1968) shows that, in steady-state, the time between the arrival of parts to subsequent resource in the flow line is the same as the arrival process. Hence, the mean time between arrivals does not change and remains constant throughout the flow line. Unfortunately, since general service time distributions are allowed, the variability of the arrival time does change.

To estimate the change in variability, Marshall (1968) explored the arrival time variability for subsequent resources of a tandem queuing system and defined a formula for estimating the variance of the interdeparture interval (output process). Rewriting this formula in terms of the flow line terminology results in the following estimator of the arrival variability to a resource:

$$E[v_{i+1}] = v_i + 2s_i^2 - \frac{2T_i}{\lambda}(1-\rho_i)$$

(1)

where: $E[v_{i+1}]$ Variability of the output process of resource R_{i+1} (i = 1,...,N)

                v_i Variability of the arrival process to resource R_i (i = 2,...,N)

                s_i^2 Variability of resource R_i's service time (i = 1,...,N)

                ρ_i Traffic intensity at resource R_i: $\rho_i = \dfrac{\lambda}{\mu_i}$ (i = 1,...,N)

                λ Arrival rate to the flow line

                μ_i Average service rate at resource R_i (i = 1,...,N)

                T_i Expected waiting time for resource R_i (i = 1,...,N) using the G/G/1 formula.

Since parts arrive to the flow line according to a Poisson process, v_1, the arrival variability to the first resource, R_1, is always equal to $1/\lambda^2$.

Using the M/G/1, G/G/1 and variance estimating formulas allows for the estimation of the cycle time of each resource. With these estimates, the average cycle time $\left(\overline{T}^* \right)$ represented by the aggregation resource can be defined as follows:

$$E\left[\overline{T}^* \right] = \frac{\sum_{j=1}^{N} T_j}{N}$$

Thus, the average cycle time of an aggregation resource is the sum of all resource cycle times aggregated by the aggregation resource divided by the number of resources aggregated.

3 AGGREGATION RESOURCE SERVICE MEAN

The second step in determining the distribution weight is to compute the service mean needed to model an aggregate resource with the given average aggregate cycle time and arrival rate. The procedure for accomplishing this involves applying queueing formulas backwards, in that the mean service time of an aggregation resource is estimated from the average cycle time. Using an M/G/C queueing formula (Hokstad 1978; Stoyan 1976) and solving for δ_1^* generates the following estimator for the aggregation resource service mean:

$$E\left[\delta_1^* \right] = \frac{\left(1 + \lambda \overline{T_1^*} \right) \pm \sqrt{1 + 2\lambda cv_{\delta_1^*}^2 \, \overline{T_1^*} + \lambda^2 \, \overline{T_1^*}^2}}{\lambda - \lambda cv_{\delta_1^*}^2} \tag{2}$$

where: $E\left[\delta_1^* \right]$ Mean service time of aggregate resource one (AR$_1$)

  λ Arrival rate of parts to the flow line

  $cv_{\delta_1^*}^2$ Squared coefficient of variation of the unknown service time δ_1^*

With values for $\overline{T_1^*}$ (the average aggregate resource cycle time) and λ (the arrival rate) known, the only unknown in the above equation is the squared coefficient of variation ($cv_{\delta_1^*}^2$) of the aggregate resources service mean (δ_1^*). Since the aggregate resource service mean is unknown (it is the quantity that this procedure is attempting to compute), a value of $cv_{\delta_1^*}^2$ must itself be estimated. Let the squared coefficient of variation for the aggregation resource, be a weighted average of the squared coefficient of variation of each of the service distributions aggregated by the aggregation resource. Mathematically this is:

$$E\left[cv_{\delta_1^*}^2\right] = \sum_{j=1}^{N}\left(\frac{T_j}{T_1^*}\right)\left(\frac{s_j^2}{m_j^2}\right)$$

Note that the weighting is a resource's relative contribution toward the aggregation resource's total cycle time. Using (2) and solving for the positive value of δ_1^* results in an estimate of the aggregate resource's service mean. This service mean will be used in the next section as the basis for determining the weights of the original service time distributions.

4 DISTRIBUTION WEIGHTS

To use composite sampling to represent the aggregation resource service time distribution, two conditions must be met: (1) the sum of all the resource weights multiplied by their respective original resource mean service time must equal the average service time of the aggregation resource (δ_1^*) and, (2) the weights must sum to one and be positive. More formally, these two conditions are:

$$(1)\ \sum_{j=1}^{N}w_j^*m_j = \delta_1^* \quad \text{and} \quad (2)\ \sum_{j=1}^{N}w_j^* = 1, \quad w_j^* \geq 0$$

This convex relationship determines the proportional weight that each resource service mean contributes towards the average service time of the aggregation resource.

The easiest case for which to determine distribution weights is a flow line in which the aggregation resource represents a single resource. The single resource would be called R_1. As a single resource, the aggregate service mean (δ_1^*) is merely the resource's service mean, m_1. Thus, the distribution weight for the resource service mean, w_1^* of resource R_1, is 1.0, which clearly satisfies the two weighting conditions.

Determining the distribution weights for two aggregated resources (*e.g.*, R_1 and R_2) is similarly easy. Recall that our objective in determining the weights is to decide how to weight the two individual service resource means (m_1 and m_2) in such a way that they equal the aggregate service mean (δ_1^*). Applying the two weighting conditions results in the following equations:

$$(w_1^* \times m_1) + (w_2^* \times m_2) = \delta_1^*$$
$$w_1^* + w_2^* = 1$$

Since values of m_1, m_2, and δ_1^* are known, the task of solving for w_1^* and w_2^* simply involves applying standard algebraic procedures for solving two equations with two unknowns.

Following similar logic, considers what occurs when the aggregation resource consists of three resources. To determine the distribution weights requires solving two equations with three unknowns. For example:

$$(w_1^* \times m_1) + (w_2^* \times m_2) + (w_3^* \times m_3) = \delta_1^* \tag{3}$$
$$w_1^* + w_2^* + w_3^* = 1$$

In this instance, the solution can only be reduced to a set of relationships among the variables. Determining a more specific solution requires much trial and error. Consider an aggregation resource that represents (say) ten resources. Here, the current solution technique involves solving two equations with ten unknowns (the weight for each of the ten resource service means). Quite a difficult, if not impossible task!

The technique to determine the service time weights must be expanded for those cases when three or more resources are represented by an aggregation resource. The solution is to combine the techniques of determining total cycle time and deriving the average aggregate resource service mean with a recursive algorithm to reduce (by aggregating) the N resources of the aggregation resource to only two resources. In essence, the technique aggregates within the aggregation resource to reduce the resources represented by the aggregation resource to only two. As demonstrated, determining the distribution weights for an aggregation resource representing two resources is easily derived by solving a set of two equations with two unknowns.

The complete algorithm is summarized in Savory (1993). As an illustration, consider a flow line consisting of three single-server resources (R_1, R_2, and R_3). The first step is to estimate the average cycle time represented by the aggregation resource and to compute an estimated service mean for the aggregation resource. This is illustrated in part (a) of Figure 3. As discussed previously, solving for the distribution weights in equation (3) results in having to solve a system of two equations with three unknowns. To reduce the number of resources represented by the aggregation resource, aggregate two resources (*e.g.*, R_1 and R_3) within the aggregation resource. This is done by summing the cycle time of two resources (T_1 and T_3) and dividing this by two to find the average cycle time of the "new" aggregate resource. That is, the average cycle time of

aggregate resource, $A_{1|3}$ (an aggregate resource within an aggregation resource) is $\overline{T_{1|3}^*}$, where $\overline{T_{1|3}^*}$

$= (T_1 + T_3)/2$. Next, compute the mean service time ($\delta_{1|3}^*$) for a resource with average cycle time

$\overline{T_{1|3}^*}$. This is demonstrated in part (b) of Figure 3.

<<<< **Figure 3 Approximately Here** >>>>

The aggregation step reduces the number of distinct resources represented by the aggregation resource by one (since two were aggregated together). Thus, determining the weights is reduced to solving the following two equations:

$$\left(w_{1|3}^* \times \delta_{1|3}^*\right) + \left(w_2^* \times m_2\right) = \delta_1^* \qquad (4)$$
$$w_{1|3}^* + w_2^* = 1$$

where $w_{1|3}^*$ is the weight and $\delta_{1|3}^*$ is the average service time computed for the aggregate resource resulting by aggregating R_1 and R_3. For larger problems, the aggregation process would continue until only two resources are represented by AR_1.

Since the value for m_2 is known and the values of $\delta_{1|3}^*$ and δ_1^* will have been computed, (4) can easily be solved using standard algebraic techniques for solving two equations with two unknowns. Doing so results in distribution weights which represent the proportional weight of each service time distribution to generate an aggregate service mean of δ_1^*. For instance, the value computed for w_2^* is the distribution (or percentage) weight that m_2 contributes towards an aggregate resource service mean of δ_1^*. The value for $w_{1|3}^*$ is the percentage weight of all the other (aggregated) resources of the aggregation resource.

The reason this approach has been termed recursive is that now that the problem has been reduced to a point in which it can be solved, the procedure works incrementally backwards using its current and subsequent solutions to solve the previous level of resource aggregation. For example, once (4) has been solved, (3) can be solved to find values for w_1^* and w_3^*. In a more complex example, the backward process of the algorithm would continue until all original resources represented by the aggregation resource have distribution weights. The result of applying this algorithm is a set of weights representing the relative significance of each service time distribution to be used for the composite sampling scheme.

5   SIMULATION MODEL SPECIFICATION

After determining the distribution weights, a final task it to develop the aggregate simulation model. The objective of this model is to estimate the average cycle time for a part to be processed by all stages of the tandem queuing system. By modeling the arrival process, the single production step, and the leaving process of the aggregation resource, the average cycle time can be collected by running the simulation model. The service time of the single production station uses composite random number generation structured around the original service time distributions and the computed distribution weights. Be aware that since the aggregate resource is an average of all the original resources (R_i), the true cycle time of a part through an aggregation flow line is N (the number of resources aggregated) multiplied by the average simulation estimate. The Appendix demonstrates the application of the aggregation methodology for three single-server resources in tandem.

6 PRELIMINARY RESULTS

To test the effectiveness of applying the aggregation methodology, ten single-server flow line scenarios were randomly generated by a Mathematica program (Savory 1993). Table 1 describes each of these test scenarios. For example, Scenario 1 is a flow line consisting of nine resources, with the service time distribution of the first resource being uniform and the second resource having an exponential service time distribution. The average utilization of the nine resources is 35.47%. Using the techniques of this paper, these nine single-server resources are combined into a single aggregation resource (AR_1).

The full flow line model and its aggregate equivalent was written in the SLAM II simulation language (Pritsker 1986) for each of the test scenarios. Thirty replications of each of the simulation models were run under steady-state conditions. A complete description of the service time parameters and the results from running the full and aggregate simulation models can be found in Savory (1993). Table 2 summarizes the results. The average relative error,

$$RE = 100\% \times \left[\frac{\left| \text{Average aggregate cycle time} - \text{Average full model cycle time} \right|}{\text{Average full model cycle time}} \right]$$

of the aggregate simulation model's estimate of the cycle time is only 4.8735%. A 95% confidence interval computed on the average relative error of the cycle time for the ten scenarios is: (4.3333%, 5.4137%). To explore the output variability of the cycle time estimates, Table 1 also illustrates the difference between the full and aggregate simulation model's coefficient of variation. The average difference in the variation for all scenarios is .00062164 or .06%. Overall, it appears the aggregation procedure closely estimates the average cycle time. In

addition, it appears that the variability of the output distribution generated by the aggregate and full model are similar for the single-server system.

<<<< **Table 1 Approximately Here** >>>>

7 FINAL COMMENTS

The paper presents a procedure for aggregating a single-server tandem queueing system. It proposes that all resources or stations are combined into a single processing step or aggregation resource. The aggregation process uses queueing theory to estimate the cycle time of the flow line and to find the service mean of an aggregation resource. It applies a recursive algorithm for determining the weight or relationship between each of the service time distributions. Using these weights, it uses composite random number sampling to replicate the service distribution of the aggregation resource. Testing reveals that the aggregation works well for estimating the mean and variability of the cycle time of a part and does not effect the output process of the tandem queuing system. The results allow for simulation models of tandem queueing systems to be executed more effectively.

Future areas for expansion include incorporating finite capacity waiting areas, allowing for multiple server resources, and permitting part rework or rejects. Our research currently has a limitation. The aggregation approach depends on estimating a resource's cycle time using the G/G/1 queueing formula in combination with the formula for estimating arrival variability. We correctly conclude that a Poisson arrival process to an exponential resource results in the departure process being Poisson. Hence, subsequent resources will also "see" a Poisson arrival process. Once the arrival process experiences a non-exponential resource, however, subsequent arrival processes will not only not be Poisson, they will in general also not have independent

interarrival times. Several papers (Patuwo et al, 1991; Szekli 1995; and Szekli et al. 1993), have shown that, beyond variability, correlation in the arrival process can drastically affect the occupancy and waiting time distributions. Our approach assumes independence. While this assumption is not necessarily true, we feel future research will show it is has minimal impact on our results.

ACKNOWLEDGMENTS

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APPENDIX

Consider three single-server resources in tandem. Parts arrive to the flow line following an exponential distribution with a mean time between arrivals of 100 minutes. The services time (in minutes) for each of the resources is given below:

- Resource 1 (R_1): Uniform(75,85)
- Resource 2 (R_2) : Triangular(32,43,60)
- Resource 3 (R_3): Uniform(64,80)

For example, the service distribution of resource R_2 is the triangular distribution with parameter values of 32, 43, and 60 representing the minimum, mode, and maximum, respectively.

In the aggregation representation of this flow line, resources R_1, R_2, and R_3 are aggregated together to form aggregation resource AR_1 (the aggregation of all single server resources). AR_1 is

represented by Q_1^* and machine M_1^*. The specification for representing f_1^* (the aggregate service time distribution) is the objective of the remainder of this example.

Table 2 presents the results from computing summary statistics for the three resources. For instance, resource R_1 has a mean service time of 80 minutes, and a service time variance of 8.3333 minutes^2. As such, the squared coefficient of variation is computed to be .00130208. Applying the M/G/1 queuing formula results in R_1 having an estimated cycle time (T_1) of 240.208 minutes:

$$T_1 = \frac{\lambda\left(m_1^2 + \sigma_{m_1}^2\right)}{2\left(1 - \rho_1\right)} + m_1 = \frac{1/100\left(80^2 + 8.3333\right)}{2(1-.8)} + 80 = 240.2083$$

That is, on average, a part will spend 240.208 minutes waiting for service and being service by R_1. The variability of the arrival process to R_2, v_1, can be estimated by equation (1), which computes the variability of resource R_1's output process:

$$v_1 = Var(R_1) = \sigma_{a_1}^2 + 2\sigma_{m_1}^2 - \frac{2}{\lambda}(1 - \rho_1)E[T_1]$$

$$= 100^2 + 2(8.3333) - \frac{2}{.01}(1-.8)(240.2083 - 80)$$

$$= 3608.3333$$

Thus, the arrival process to R_2 has a mean of 100 (since the mean time between arrivals remains constant throughout the flow line) and a variance of 3608.33333. Therefore, the squared coefficient of variation of the arrival variation is $\frac{3608.33333}{100^2} = .360833$. Using the G/G/1 formula, the expected cycle time of R_2 can be estimated:

$$T_2 = \left[\frac{cv_{a_2}^2 + cv_{m_2}^2}{2} g\left\{\frac{\rho_2}{1/m_2 - \lambda}\right\}\right] + m_2$$

$$= \left[\frac{.360833 + .0163786}{2}(.413758)\left\{\frac{.45}{.02222 - .01}\right\}\right] + 45$$

$$= 47.8732$$

With an estimate of the cycle time for R_2, the arrival variability to R_3 (output variability of R_2) can be computed:

$$\begin{aligned}
v_2 = Var(R_2) &= \sigma_{a_2}^2 + 2\sigma_{m_2}^2 - \frac{2}{\lambda}(1-\rho_2)E[T_2] \\
&= 3608.3333 + 2(33.1667) - \frac{2}{.01}(1-.45)(47.8732 - 45) \\
&= 3358.62
\end{aligned}$$

Thus, the arrival process to R_3 has a mean of 100 and a variance of 3358.62. The squared coefficient of variation of the arrival variation is $\frac{3358.62}{100^2} = ..335862$. Using the G/G/1 formula, the expected cycle time of R_3 can be estimated:

$$\begin{aligned}
T_3 &= \left[\frac{cv_{a_3}^2 + cv_{m_3}^2}{2} g\left\{\frac{\rho_2}{1/m_3 - \lambda}\right\}\right] + m_3 \\
&= \left[\frac{.335862 + .00411523}{2}(.714367)\left\{\frac{.72}{1.3889 - .01}\right\}\right] + 72 \\
&= 94.4827
\end{aligned}$$

With all cycle times computed, the average cycle time of the aggregation resource, AR_1, can be determined:

$$\begin{aligned}
\overline{T_1^*} &= \frac{T_1 + T_2 + T_3}{3} \\
\overline{T_1^*} &= \frac{240.2083 + 47.8732 + 94.4827}{3} = \frac{382.564}{3} = 127.521
\end{aligned}$$

<<<< Table 2 Approximately Here >>>>

Before computing the mean service time, the squared coefficient of variation of the service time for AR_1 must be estimated. This involves weighting the squared coefficient of variation of each resource's service time by the percentage of that resource's cycle time toward the overall total cycle time of the aggregation resource. For AR_1, the estimate of $cv_{\delta_1}^2$ is:

$$cv_{\delta_1}^2 = \left[\left(\frac{240.208}{382.564}\right).00130208\right] + \left[\left(\frac{47.8732}{382.564}\right).0163786\right] + \left[\left(\frac{94.4827}{382.564}\right).00411523\right]$$
$$=.00388349$$

Using this result, the mean service rate of AR_1 (δ_1^*) can be found. Solving (2) results in δ_1^* being

equal to 65.4155.

This third step of the aggregation methodology uses the mean service time of a resource to

determine its contribution towards the aggregate service time mean. Since aggregation resource

AR_1 represents three resources, determining the weights involves applying the recursive

procedure. Specifically, it is necessary to solve:

$$80w_1^* + 45w_2^* + 72w_3^* = 65.4155$$
$$w_1^* + w_2^* + w_3^* = 1$$

The first task in applying the recursion is to aggregate two of the resources within the aggregation

resource. Thus, aggregating (say) R_1 and R_3 yields a new aggregation resource: $AR_{1|3} = \{R_1, R_3\}$.

Logic in determining which resources to aggregate is presented in Savory (1993). The total cycle

time of this aggregate resource is:

$$T_{1|3}^* = T_1 + T_3 = 240.208 + 94.4827 = 334.69070$$

The average cycle time of $AR_{1|3}$ is:

$$\overline{T_{1|3}^*} = \frac{T_{1|3}}{2} = \frac{334.69070}{2} = 167.34535$$

To determine the mean service time needed to generate an average cycle time of 167.34535

requires estimating the squared coefficient of variation:

$$cv_{1|3}^{2^*} = \left[\left(\frac{240.208}{334.69070}\right).00130208\right] + \left[\left(\frac{94.4872}{334.69070}\right).00411523\right]$$
$$=.002090228$$

Using this value, the mean service time of $AR_{1|3}$ is computed to be $\delta_{1|3}^* = 72.3469$.

Now that R_1 and R_3 have been aggregated, the explicit number of resources represented by aggregation resource AR_1 is reduced to only $AR_{1|3}$ and R_2. Thus, the aggregation resource represents two resource, $AR_{1|3}$ which has a service mean of 72.3469 and R_2 with a service mean of 45. With only two resources represented, the weights can be determined:

$$72.3469w_{1|3}^* + 45w_2^* = 65.4155$$
$$w_{1|3}^* + w_2^* = 1$$

Solving yields $w_{1|3}^* = .746538$ and $w_2^* = .253462$. Thus, the contribution of m_2 towards the aggregation resource service time is 25.3462%, while the other (currently aggregated) resources contribute 74.6538%.

With w_2^* known, the next step is to go to the previous level of aggregation and plug this value into the equations:

$$80w_1^* + 45(.253462) + 72w_3^* = 65.4155$$
$$w_1^* + .253462 + w_3^* = 1$$

These equations reduce to:

$$80w_1^* + 72w_3^* = 54.00971$$
$$w_1^* + w_3^* = .746538$$

Solving yields the values: $w_1^* = .0323717$ and $w_3^* = .714166$. Note that the sum of w_1^*, w_2^*, and w_3^* is 1.00. These weights will next be used to develop the aggregate simulation model of the flow line system. Each weight will represent the weight of the resource service time distribution in estimating the aggregation resource service time distribution.

The final task is to specify the composite random number sample schemes for representing AR_1. Recall that the resources R_1, R_2, and R_3 are characterized by their service time distributions, f_1 = Uniform(75,85), f_2 = Triangular(32, 43, 60), and f_3 = Uniform(64, 80), and

their distribution weights, $w_1^* = .0323717$, $w_2^* = .253462$, and $w_3^* = .714166$. Hence, the composite sampling distribution for representing AR_1 is:

$$f_1^*(I) = \begin{cases} Uniform(75,85) & 0 \le I < .0323717 \\ Triangular(32,43,60) & .0323717 \le I < .2858337 \\ Uniform(64,80) & .2858337 \le I \le 1 \end{cases}$$

where I is a Uniform(0,1) random number that is generated when a sample from f_1^* is needed. Figure 4 displays a subset of the SLAM II simulation model for representing this example. Note that ATRIB(3) records the service time and ATRIB(4) records the *average* cycle time. The final attribute, ATRIB(5), records the *total* cycle time by multiplying the *average* cycle time by three to account for the fact that three resources were aggregated by the aggregation resource.

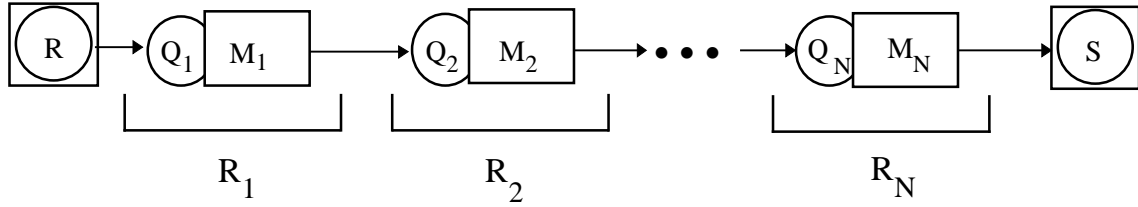**<<<< Figure 4 Approximately Here >>>>**

Figures

Figure 1. A tandem queueing system consisting of N resources.
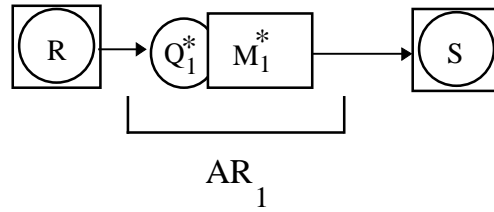

Figure 2. An aggregate representation of a single-server tandem queueing system.
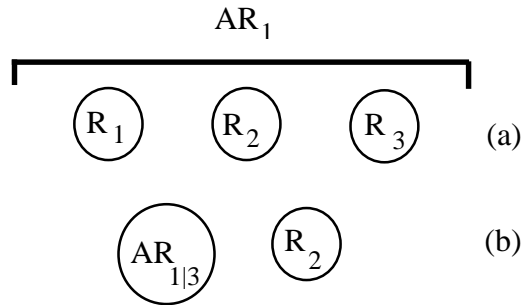

Figure 3. Example of the recursive procedure to determine the distribution weight for an aggregation resource consisting of three resources.

```
        CREATE,EXPON(100),,1;
;
        GOON,1;
            ACT,,.0323717,A11;
            ACT,,.2534620,A12;
            ACT,,.7141660,A13;
A11 ASSIGN, ATRIB(3)=UNFRM(75,85);
            ACT,,,D1;
A12 ASSIGN, ATRIB(3)=TRIAG(32,43,60);
            ACT,,,D1;
A13 ASSIGN, ATRIB(3)=UNFRM(64,80);
            ACT,,,D1;
;
D1  Queue(1);
            ACT(1)/1,ATRIB(3);
;
        ASSIGN,ATRIB(2)=TNOW-ATRIB(1)-ATRIB(3);
        ASSIGN,ATRIB(4)=ATRIB(2)+ATRIB(3);
        COLCT,ATRIB(3),AR1 SERVICE TM;
        COLCT,ATRIB(4),AR1 CYCLE TM;
        ASSIGN,ATRIB(5)=ATRIB(4)*3;
        COLCT,ATRIB(5),AR1 TOTAL CYCLE;
```

Figure 4. SLAM II code for modeling the aggregation resource representing the three tandem resources.

Tables

Table 1: Test case scenarios. The service time distributions are: UN = uniform, EX = exponential, LN = lognormal, TR = triangular, RN = normal. The average utilization for the test scenarios is given in Average Utilization. Relative Error is the relative difference between comparing the aggregate simulation model estimate of cycle time to the full model simulation results. The Difference of CV measures the difference in the coefficient of variation of the cycle time estimates.

Scenario	Number of Resources	Service time Distributions of Resources flow line (in order)	Average Utilization	Relative Error	Difference of CV
1	9	UN, EX, LN, UN, TR, RN, RN, TR, EX	.3547	0.0931%	-.0000014
2	8	EX, RN, LN, RN, EX, LN, RN, EX	.4881	2.0166%	.0019032
3	8	TR, EX, TR, LN, EX, EX, TR, TR	.5902	4.6510%	.0014228
4	8	RN, RN, LN, LN, EX, EX, UN, UN	.4210	3.6070%	.0015125
5	7	UN, EX, RN, RN, TR, UN, UN	.4731	9.0020%	.0001701
6	6	TR, LN, UN, EX, UN, LN	.5315	6.8706%	.0011038
7	8	TR, EX, TR, EX, LN, UN, RN, TR	.4231	7.3859%	-.0000071
8	7	TR, TR, RN, TR, EX, UN, TR	.3484	5.6569%	.0012286
9	5	TR, UN, TR, RN, LN	.3874	7.5506%	.0004253
10	10	TR, TR, RN, LN, RN, EX, TR, EX, TR, UN	.5228	1.9013%	-.0014644

Table 2: Summary statistics for the three resources of the flow line system. The variance, squared coefficient of variation, and cycle time has been computed for each of the resources.

	R_1	R_2	R_3
Mean (m_j) =	80	45	72
Variance ($\sigma^2_{m_j}$) =	8.3333	33.1667	21.3333
Square COV ($cv^2_{m_j}$) =	.00130208	.0163786	.00411523
Arrival Mean =	100	100	100
Arrival Variability (v_i) =	$100^2 = 10{,}000$	3608.3333	3358.62
Est. Cycle Time (T_i) =	240.208	47.8732	94.4827