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A PROOF OF NEWTON'S POWER SUM FORMULAS

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For a polynomial $P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n = \alpha_n (z - z_1)(z - z_2) \cdots (z - z_n)$, the power sums $S_m = \sum_{k=1}^n z_k^m$, $m = 1, 2, \cdots$, can be calculated from the formulas

$$(1) \quad m\alpha_{n-m} + \sum_{k=1}^m \alpha_{n-m+k} S_k = 0 \quad \text{if } m \leq n,$$

$$\sum_{k=m-n}^m \alpha_{n-m+k} S_k = 0 \quad \text{if } m > n.$$

For example, if $n = 3$,

$$S_1 = -\alpha_2 \alpha_3^{-1}, \quad S_2 = \alpha_2^2 \alpha_3^{-2} - 2\alpha_1 \alpha_3^{-1}, \quad S_3 = -\alpha_2^3 \alpha_3^{-3} + 3\alpha_1 \alpha_2 \alpha_3^{-2} - 3\alpha_0 \alpha_3^{-1},$$

$$S_4 = \alpha_2^4 \alpha_3^{-4} - 4\alpha_1 \alpha_2^2 \alpha_3^{-3} + 4\alpha_0 \alpha_2 \alpha_3^{-2} + 2\alpha_1^2 \alpha_3^{-2}.$$

One quick, but rather vague, method of proving (1) is to differentiate $P(z)$ two different ways, equating like powers of z (see [1]). Another method is through the use of the theory of symmetric functions (see [2]). The student might find the following proof more satisfying: The logarithmic derivative of the polynomial $Q(z) = \alpha_n + \alpha_{n-1}z + \cdots + \alpha_0 z^n = \alpha_0(z - z_1^{-1})(z - z_2^{-1}) \cdots (z - z_n^{-1})$ (assuming, without loss of generality, that $P(0) \neq 0$) is

$$F(z) = \frac{Q'(z)}{Q(z)} = \sum_{k=1}^n (z - z_k^{-1})^{-1}$$

which, when differentiated k times, gives $F^{(k)}(0) = -k!S_{k+1}$. Since

$$Q^{(m)}(z) = [F(z)Q(z)]^{(m-1)} = \sum_{k=0}^{m-1} \binom{m-1}{k} F^{(k)}(z)Q^{(m-1-k)}(z),$$

we have

$$\frac{Q^{(m)}(0)}{m!} = -\frac{1}{m} \sum_{k=0}^{m-1} \frac{Q^{(m-1-k)}(0)}{(m-1-k)!} S_{k+1}$$

or

$$\begin{aligned} -m\alpha_{n-m} &= \sum_{k=0}^{m-1} \alpha_{n-m+k+1} S_{k+1} && \text{if } m \leq n, \\ 0 &= \sum_{k=m-n-1}^{m-1} \alpha_{n-m+k+1} S_{k+1} && \text{if } m > n, \end{aligned}$$

which is (1).

References

1. L. E. Dickson, *Linear Groups*, B. G. Teubner, Stuttgart, 1901, p. 53.
2. B. L. van der Waerden, *Modern Algebra*, Frederick Ungar, New York, 1953.