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A PROOF OF NEWTON'S POWER SUM FORMULAS

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For a polynomial $P(z)=\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n=\alpha_n(z-z_1)(z-z_2)\cdots(z-z_n)$, the power sums $S_m=\sum_{k=1}^n z_k^m$, $m=1, 2, \cdots$, can be calculated from the formulas

$$
(1)\qquad
\begin{aligned}
m\alpha_{n-m}+\sum_{k=1}^{m}\alpha_{n-m+k}S_k &= 0 \quad \text{if } m \leqq n,\\
\sum_{k=m-n}^{m}\alpha_{n-m+k}S_k &= 0 \quad \text{if } m > n.
\end{aligned}
$$

For example, if $n=3$,

$$S_1 = -\alpha_2\alpha_3^{-1},\qquad S_2=\alpha_2^2\alpha_3^{-2}-2\alpha_1\alpha_3^{-1},\qquad S_3=-\alpha_2^3\alpha_3^{-3}+3\alpha_1\alpha_2\alpha_3^{-2}-3\alpha_0\alpha_3^{-1},$$
$$S_4=\alpha_2^4\alpha_3^{-4}-4\alpha_1\alpha_2^2\alpha_3^{-3}+4\alpha_0\alpha_2\alpha_3^{-2}+2\alpha_1^2\alpha_3^{-2}.$$

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One quick, but rather vague, method of proving (1) is to differentiate $P(z)$ two different ways, equating like powers of z (see [1]). Another method is through the use of the theory of symmetric functions (see [2]). The student might find the following proof more satisfying: The logarithmic derivative of the polynomial $Q(z)=\alpha_n+\alpha_{n-1}z+\cdots+\alpha_0 z^n=\alpha_0(z-z_1^{-1})(z-z_2^{-1})\cdots(z-z_n^{-1})$ (assuming, without loss of generality, that $P(0)\neq 0$) is

$$F(z) = \frac{Q'(z)}{Q(z)} = \sum_{k=1}^{n} (z - z_k^{-1})^{-1}$$

which, when differentiated k times, gives $F^{(k)}(0)=-k!S_{k+1}$. Since

$$Q^{(m)}(z) = [F(z)Q(z)]^{(m-1)} = \sum_{k=0}^{m-1} \binom{m-1}{k} F^{(k)}(z)Q^{(m-1-k)}(z),$$

we have

$$\frac{Q^{(m)}(0)}{m!} = -\frac{1}{m}\sum_{k=0}^{m-1} \frac{Q^{(m-1-k)}(0)}{(m-1-k)!} S_{k+1}$$

or

$$-m\alpha_{n-m} = \sum_{k=0}^{m-1} \alpha_{n-m+k+1}S_{k+1} \qquad \text{if } m \leqq n,$$

$$0 = \sum_{k=m-n-1}^{m-1} \alpha_{n-m+k+1}S_{k+1} \quad \text{if } m > n,$$

which is (1).

References

1. L. E. Dickson, Linear Groups, B. G. Teubner, Stuttgart, 1901, p. 53.
2. B. L. van der Waerden, Modern Algebra, Frederick Ungar, New York, 1953.