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Embedding and Nonembedding Results for R. Thompson's Group V and Related Groups

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EMBEDDING AND NONEMBEDDING RESULTS FOR R. THOMPSON'S GROUP V AND RELATED GROUPS

by

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EMBEDDING AND NONEMBEDDING RESULTS FOR R. THOMPSON'S GROUP V AND RELATED GROUPS

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We study Richard Thompson's group V, and some generalizations of this group. V was one of the first two examples of a finitely presented, infinite, simple group. Since being discovered in 1965, V has appeared in a wide range of mathematical subjects. Despite many years of study, much of the structure of V remains unclear. Part of the difficulty is that the standard presentation for V is complicated, hence most algebraic techniques have yet to prove fruitful.

This thesis obtains some further understanding of the structure of V by showing the nonexistence of the wreath product $\mathbb{Z} \wr \mathbb{Z}^2$ as a subgroup of V, proving a conjecture of Bleak and Salazar-Dìaz. This result is achieved primarily by studying the topological dynamics occurring when V acts on the Cantor Set. We then show the same result for one particular generalization of V, the Higman-Thompson Groups $G_{n,r}$. In addition we show that some other wreath products do occcur as subgroups of nV, a different generalization of V introduced by Matt Brin.

DEDICATION

Dedicated to Amy Cohen Corwin and to the memories of Leon Cohen, Bernard Corwin, and Lawrence Corwin.

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Chapter 1

Introduction

In broad strokes, this thesis concerns topics in geometric group theory. This area studies infinite groups, in particular groups that are finitely generated. Although it has roots in a publication by William Rowan Hamilton in 1856, geometric group theory became an established field upon the 1987 publication of "Hyperbolic groups" by Mikhail Gromov. It has become a field in the intersection of group theory and topology with interactions with many other areas of mathematics along with some parts of theoretical computer science.

More specifically, this thesis contributes to the study of Thompson's group V, and some generalizations of this group, by studying the structure of these groups through showing the existence or lack of certain subgroups. The groups $F < T < V$ were discovered by Richard Thompson in 1965 during research into mathematical logic. A standard introduction to the area is [13].

Thompson's group V can be thought of as the subgroup of the automorphism group of the Cantor set $\mathfrak{C}$ in which the functions are completely determined by a finite list of prefix replacements. Interest in this group is broad as V appears in a variety of mathematical subjects, including the theory of interval exchange maps from dynamics [1], algorithmic group theory [19], circuit and complexity theory [3], and logic and combinatorial group theory [25].

In group theory, the first evidence that these groups are of interest came from Thompson himself soon after his discovery of the group. In unpublished notes, Thompson proved that T and V are finitely presented simple infinite groups. These were the first two known examples of such groups. More recently, the conjugacy problem has been solved in V in three separate, but related, ways [2, 16, 22], and many of the homological properties of V have been investigated [12]. In [19] it is shown that V has a context free co-word problem. There are other structure results, for example [5, 8]. However, it appears that using an algebraic approach using a presentation is hard, and not too much is known about V. This is in contrast to F which is well understood and many results have been published using purely algebraic methods.

Context free languages arise in formal language theory. A subset of the set of all strings from a fixed alphabet is *context free* if there exists a non-deterministic finite state pushdown automata (NPDA) that recognizes the language. Recall that for a group G, the *word problem for G* is the set of all strings in a presentation that are equal in the group to the identity. It was shown by Muller and Schupp in [20, 21] that the class $\mathscr{CF}$ of groups that have a context free word problem is equivalent to the the class of groups that are virtually free (i.e., those that have a finite index subgroup which is free). A natural generalization of the class $\mathscr{CF}$ is the class $co\mathscr{CF}$, groups that have context free co-word problem, and was first introduced by Holt, Rees, Röver, Thomas in [17]. A group is in $co\mathscr{CF}$ when the set of all strings in a presentation that are not equal to the identity is context free. Both of these classes are independent of the choice of finite generating set used.

In [17], it is shown that $co\mathscr{CF}$ is closed under direct products, standard restricted wreath products where the top group is $\mathscr{CF}$, passing to finitely generated subgroups, and passing to finite index over-groups. We are primarily concerned with two conjectures from that paper. The first is if $C \wr T$ is in $co\mathscr{CF}$ then T must be in $\mathscr{CF}$. The second is that $co\mathscr{CF}$ is not closed under free product. The most popular candidate to show the latter is $\mathbb{Z} * \mathbb{Z}^2$. This was put into doubt by Lehnert and Schweitzer in [19] which shows V, which is rife with free products of its subgroups and contains abundant copies of $\mathbb{Z}$ and $\mathbb{Z}^2$, is in $co\mathscr{CF}$. However, Bleak and Salazar-Dìaz showed in [8] that $\mathbb{Z} * \mathbb{Z}^2$ does not embed into V.

One interesting aspect of the proof in [8] is that it suggests that V does not have a subgroup that acts on the Cantor Set locally as $\mathbb{Z}^2$. This would be necessary to build the desired free product using a standard ping-pong argument. It would also be necessary to build the wreath product $\mathbb{Z} \wr \mathbb{Z}^2$ using Brin's pre-wreath structure construction.

In [4] Bleak showed that $\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into F, and in [6] Bleak, Kassabov, and Matucci showed $\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into T. Guba and Sapir showed in [15] that there are subgroups isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ in F, and hence in T and V as well.

In Section 2, we provide necessary background. In Section 3, we analyze the dynamics of V acting on the Cantor set to show the following:

Theorem 19. *$\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into V.*

This proves Bleak and Salazar-Dìaz's Conjecture 4 in [8]. Additionally, as $\mathbb{Z}^2$ is not virtually free, the theorem adds some evidence to the conjecture that the top group must be $\mathscr{CF}$.

Section 4 concerns some generalizations of V. The best known generalization are the finitely presented simple groups $G_{n,r}$ collectively known as the Higman-Thompson groups. We show the following:

Corollary 21. *$\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into $G_{n,r}$ for any pair of integers $r \geq 1$ and*

$n \geq 2$.

A different generalization V is the collection of groups nV originally described by Brin in [9]. These are also finitely presented simple groups with $V = 1V$. It is immediate from the definition of nV that if $n < m$ then $nV < mV$. It was shown in [7] that mV is not isomorphic to nV when $m \neq n$. This proof uses a theorem of Rubin. This result does not seem to be easily modified to show the following.

Conjecture 1. *If $n < m$ then mV does not embed into nV.*

The main obstacle to adapting the approach in [7] is that while the *locally dense* property of a group of homeomorphisms is preserved under isomorphisms, it is not preserved under general group monomorphisms. This property is needed to apply Rubin's Theorem.

One possible approach to showing Conjecture 1 is to find groups H_n such that H_n injects into nV but H_n does not inject into $(n-1)V$. To this end, we make the following two conjectures.

Conjecture 2. *For $n \in \mathbb{N}$, $\mathbb{Z} * \mathbb{Z}^{n+1}$ does not embed into nV.*

Conjecture 3. *For $n \in \mathbb{N}$, $\mathbb{Z} \wr \mathbb{Z}^{n+1}$ does not embed into nV.*

In this paper, we show two additional results that deal with the existence of some embeddings into nV. These, combined with a proof of either Conjecture 2 or Conjecture 3 would give a proof of Conjecture 1.

Theorem 25. *For $n \in \mathbb{N}$, $\mathbb{Z} * \mathbb{Z}^n$ embeds into nV.*

Theorem 27. *For $n \in \mathbb{N}$, $\mathbb{Z}^n \wr \mathbb{Z}^n$ embeds into nV.*

We desire only that $\mathbb{Z} \wr \mathbb{Z}^n$ embeds, but it takes minimal extra effort to prove the stronger statement.

The techniques used in this thesis depend heavily on the groups in the wreath products being infinite. However, it seems likely that the lack of an injection in Theorem 19 comes from the $\mathbb{Z}^2$ component. Therefore, we ask the following question.

Question 4. *If there is an injection from $G \wr \mathbb{Z}^2$ into V, must G be the trivial group?*

Chapter 2

Background Definitions

Throughout this thesis, we will have functions act on their underlying set on the right, so if $\phi : X \to X$ is a function, and $x \in X$, then $x\phi$ is the image of x under the action of ϕ. If $Y \subseteq X$, then we similarly denote by $Y\phi$ the set $\{y\phi | y \in Y\}$. Following these conventions, if additionally $\psi : X \to X$ is a bijective map, then conjugation and the commutator will be denoted as $\phi^\psi = \psi^{-1}\phi\psi$ and $[\phi, \psi] = \phi^{-1}\psi^{-1}\phi\psi = \phi^{-1}\phi^\psi = {\psi^{-1}}^\phi\psi$ respectively.

We will define Supp $(\phi) = \{x \in X | x\phi \neq x\}$. Note this is slightly different from the standard analysis definition which takes the closure of this set. That this is preferable for our purposes is evident in the following standard lemma from permutation group theory.

Lemma 5. *Let $\phi, \psi \in Aut(X)$, then Supp $(\phi^\psi) = ($Supp $(\phi))\psi$.*

2.1 Cantor Set

In this section, we fix our notation for the Cantor Set.

Let A be any set. Then A^* will denote the free monoid on A. In other words, A^*

is the set of all finite sequences with entries in A. The operation is concatenation. The set of all countable infinite sequences with entries in A will be denoted by A^ω. If $x \in A^*$ and $y \in A^\omega$ then $xy \in A^\omega$ will be the concatenation of x with y. The *Cantor Set* $\mathfrak{C}$ is a topological space with the underlying set being the set of all infinite sequences of zero's and ones i.e, $\mathfrak{C} = \{0,1\}^\omega$. We now describe the topology on $\mathfrak{C}$. Let $b \in \{0,1\}^*$ and define *the cone set of* b by $B_b = \{x \in \mathfrak{C} | x = by$ for some $y \in \mathfrak{C}\}$. The basis of $\mathfrak{C}$ is the set of cone sets for all such finite sequences.

We will be interested in open sets which are a finite union of basis elements, which we will call *conventional open sets.*

2.2 Thompson's Group V

Let $\mathcal{T}_2 = \mathcal{T}$ be the infinite, rooted, directed, binary tree. If we consider the left edge out of a node as representing 0 and the right edge representing 1, then the limit space of directed paths in $\mathcal{T}$ is exactly $\mathfrak{C}$. Any vertex in $\mathcal{T}$ can be uniquely denoted by a string $v \in \{0,1\}^*$ that gives the path from the root to the particular vertex. If the path to v passes through w we say v is a *descendent* of w, and that w is an *ancestor* of v. If the vertices are adjacent, we say v is the *parent* of $v0$ and $v1$, and the latter are the *children* of v. If $x \in \mathfrak{C}$, $v \in \{0,1\}^*$, and $x = vu$, then we say x is under v. The set of all $x \in \mathfrak{C}$ under v is called *the Cantor set under* v and is denoted as $\mathfrak{C}_v$. Note that $\mathfrak{C}_v$ is precisely the set B_v.

R. Thompson's group V can be thought of as a particular subgroup of $Aut(\mathfrak{C})$. To describe this subgroup, we need a little more terminology. Call a subgraph of $\mathcal{T}$ that consists of any vertex, its two children, and the edges between them a *caret.* Let D and R be finite rooted binary subtrees of $\mathcal{T}$, with the same root as $\mathcal{T}$, each the union of exactly n distinct carets. Each will have exactly $n+1$ leaves. An element of V can be represented (non-uniquely) by the triple (D, R, σ) where σ is a bijection from the leaves of D to the leaves of R. We will call (D, R, σ) a *tree pair representative*.

We now describe how to interpret a tree pair as an automorphism of $\mathfrak{C}$. Say that $\phi \in Aut(\mathfrak{C})$. We will use the notation $\phi \sim (D, R, \sigma)$ to denote that ϕ is represented by the tree pair (D, R, σ). Given $x \in \mathfrak{C}$, we need to identify the element of the Cantor set denoted by $x\phi$. There is a unique leaf w of D such that x is under w. Say $x = wv$. Let $w' = w\sigma$, a leaf in R. Then, $x\phi = w'v$. In other words, ϕ acts on x by replacing the prefix w of x which appears as a leaf in D with the prefix w' which appears as the leaf in R. With this interpretation, it is immediate that (D, R, σ) is a bijection from $\mathfrak{C}$ to itself. This function is continuous as the inverse image of a cone set of appropriate depth is again a cone set. It is a homeomorphism as (R, D, σ^{-1}) is an inverse. We also note that a tree pair induces an action on almost all nodes of $\mathcal{T}$. Thus, if $b \in \{0, 1\}^*$, we can often talk about $b\phi$. This only fails to have meaning on a finite number of vertices near the root of D. Note that as not every vertex or edge of $\mathcal{T}$ close to the root is mapped under the action of ϕ, ϕ is not a tree homomorphism. We will often use the phrase *maps to* when referring to the action of ϕ on either the vertices of $\mathcal{T}$ or the elements of $\mathfrak{C}$; the context should always make it clear which usage we mean.

Figure 2.1 shows an example of a tree pair $u \sim (D, R, \sigma)$, with D on the left, R on the right, and the bijection denoted by corresponding numbers on the leaves. As an example of the action of u on the leafs of D, we note $000u = 1110$ and $0101u = 00111$.

A tree pair is not a unique representative of an element. This is easily seen by taking any leaf of the domain of any tree pair and splitting it into two more leafs below. Do the same to the corresponding leaf of the range tree. Associate the new leaves to each other with the same orientation as before. This is now a new tree pair representative for the same element of V.

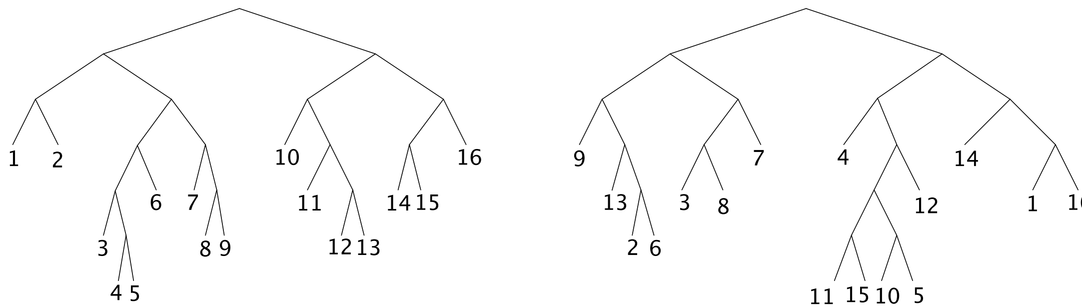


Figure 2.1: A tree pair for the element u

2.3 Revealing Pairs

Brin, in [9], showed that certain tree pair representatives - called *Revealing Pairs* - of a particular element of V can be used to find useful elements and subsets of the Cantor Set. Fix a particular tree pair (D, R, σ) representing $\phi \in V$. Consider the common tree $C = D \cap R$. This is also a finite rooted subtree of $\mathcal{T}$ consisting of the vertices and edges common to both D and R. A leaf of C is exactly one of the following:

1. the root of a non-empty subtree of D;
2. the root of a non-empty subtree of R;
3. a leaf of both D and R.

A leaf satisfying the last condition is called a *neutral leaf*. If a leaf of C satisfies the first condition, call the maximal subtree of D which is rooted at that node a *component of* $D \setminus R$. Similarly, each maximal non-empty subtree of R which is rooted at a node of C is called a *component of* $R \setminus D$. Note that we are using the phrase *component of* $D \setminus R$ to represent the phrase "topological component of the closure of $D \setminus R$ in D", where we think of D as a 1-complex. In particular, we are including the root of the maximal subtree in the component of $D \setminus R$. The analogous is true for the other case as well.

The tree pair P is called a *Revealing Pair* if it satisfies two conditions. The first condition is that for each component X of $D \setminus R$, X has a leaf r_X which, as a vertex of $\mathcal{T}$, and under iteration of the action of ϕ on the nodes of $\mathcal{T}$, travels through the neutral leaves of C until it is finally mapped to the root of X. The vertex r_X is unique for X, and is called the *repelling leaf* of X or the *repeller* of X. The other leaves of X are called *sources*.

The second condition is the similar condition for $R \setminus D$; if n_Y is the root of a component Y of $R \setminus D$ then iteration of the action of ϕ on the vertices of $\mathcal{T}$ has n_Y travel through the neutral leaves of C until it finally maps to a leaf ℓ_Y of Y. Note that ℓ_Y is a descendent of n_Y. We call the leaf ℓ_Y the *attracting leaf* of Y or the *attractor* of Y. The other leaves of Y are called *sinks*.

Brin, in the discussion proceeding Lemma 10.2 of [9], shows that each element in V has a revealing pair representative. Again, this representative is not unique. The theory of revealing pairs was expounded upon by Salazar-Díaz in [23].

Let $\phi \in V$. One consequence of Proposition 10.1 of [9] is that there is a minimal non-negative power k so that ϕ^k acts on $\mathfrak{C}$ with no non-trivial finite orbits. Set $\varphi = \phi^k$. Assume that the tree pair $P = (D, R, \sigma)$ is a revealing pair representing φ. We obtain a list of useful results that appear to have first been shown in [9] and first explicitly stated in [6].

Lemma 6. *(Brin 04; Bleak and Salazar-Dìaz 2009) Suppose $\theta \in V$ such that θ admits no non-trivial finite orbits in its action on $\mathfrak{C}$. Suppose further that a revealing pair $P_\theta = (D_\theta, R_\theta, \sigma_\theta)$ represents θ.*

1. *Any repeller r_X of a component X of $D_\theta \setminus R_\theta$ always maps to the root of X*

under P_θ .

2. *The root n_Y of any component Y of $R_\theta \setminus D_\theta$ always maps to the attractor ℓ_Y of Y under P_θ.*

3. *The map θ restricted to any Cantor set underlying a node r_X or n_Y as above is affine with slope not equal to one.*

4. *Every point in $\mathfrak{C}$ which is fixed by θ and which does not underlie a node r_X or n_Y as above lies under a neutral leaf of P_θ upon which θ must act as the identity.*

We return to our discussion of the element φ constructed previously. By an application of the standard Contraction Lemma, we observe that if a leaf ℓ of D is mapped below itself in $\mathcal{T}$ by the rule P, then there will be a unique fixed point in the Cantor Set underlying ℓ. Similarly, if ℓ maps above itself, considering the inverse map φ^{-1}, we also have a fixed point. Fixed points underlying repellers of D will be called *repelling fixed points* of φ, and fixed points underlying attractors will be called *attracting fixed points* of φ. This is summed up in the following.

Corollary 7. *(Brin 04) Suppose $\theta \in V$ such that θ admits no non-trivial finite orbits. Suppose further that a revealing pair $P_\theta = (D_\theta, R_\theta, \sigma_\theta)$ represents θ. For each repeller r_x of a component X of $D_\theta \setminus R_\theta$, there is a unique repelling fixed point p_x underlying it, and for each attractor ℓ_Y of $R_\theta \setminus D_\theta$ there is a unique attracting fixed point underlying it.*

We will be very interested in the repelling and attracting fixed points of an element throughout. Elements of the set of repelling and attracting fixed points under the action of φ are called *important points* of φ, and we denote the set of all such points as $I(\varphi)$. Note the number of important points is the same as the sum of the number of repellers and attractors which is bounded by the sum of the number of leafs of

Domain and Range trees. In particular, the cardinality of $I(\varphi)$ is finite. Important points were first defined by Bleak and Salazar-Díaz in [8].

In our example $u \in V$ from Figure 1, we see that (D, R, σ) is a revealing pair. Below are the leaves of D and R sorted into the various categories listed above.

- The neutral leaves are $000, 0101, 100$, and 1010.
- There are 4 repellers for u: $01000, 0110, 10110$, and 1100.
- There are 6 sources: $010010, 010011, 01110, 01111, 10111$, and 1101.
- There are 3 attractors: $00110, 101000$, and 1111.
- There are 6 sinks: $0010, 00111, 101001, 101010, 101011$, and 1110.
- The attracting fixed points are at $010\overline{00}, 011\overline{0}, 1011\overline{0}$, and $1011\overline{0}$.
- The repelling fixed points are at $001\overline{10}, 1010\overline{00}$, and $111\overline{1}$.

Throughout the remainder of this thesis, if we discuss the important points of an element θ of V, it is to be understood that θ does not admit finite non-trivial orbits in its action on $\mathfrak{C}$. Lastly, we note two definitions we will use in a way that is nonstandard in topological dynamics. Given a revealing pair $Q = (S, T, \rho)$ representing θ, the Cantor Set underlying each root of a component of $S \setminus T$ represents a *repelling basin* for θ, and, similarly, the Cantor set underlying a root of a component of $T \setminus S$ represents an *attracting basin* for θ.

2.4 Components of Support

We will make use of a small part of the idea of flow graphs introduced in [5]. Again, let $P = (D, R, \sigma)$ be a revealing pair representative for an element $\phi \in V$. Let $\{E_i\}$

be the components of $D \setminus R$ and $\{F_j\}$ be the components of $R \setminus D$.

As discussed in the last subsection, each component E_i has a repeller, and all other leaves are sources. Similarly, each component F_j has one leaf called an attractor, and all other leaves are called sinks. For each source leaf s_0, there is a path $s_0 = n_0, n_1, \ldots, n_t = s_k$ through neutral leaves $n_1, ..., n_{t-1}$ of C, and then visiting a sink s_k, so that ϕ^p will throw the Cantor set underlying $s_0 = n_0$ onto the Cantor set underlying n_p for all indices $0 \leq p \leq t$. Call the path $n_0, \ldots, n_t$ the *source-sink chain* $s_0 - s_k$ for P.

For our purposes, the *flow graph* of P is a labeled bipartite graph with one vertex corresponding to each repelling or attracting basin of P. The vertex on the flow graph is labeled by the root of the basin. The edges correspond with, and are labeled by, source-sink chains connecting repelling basins to attracting basins. (This is a simplified version of the flow graph as defined by Bleak et al in [5]. See that, or [8], for the more general situation.)

Let Z be a connected component of the flow graph for P. The union of the Cantor sets underlying all the vertices in the labels of Z is called *the Cantor set underlying* Z. Different revealing pairs of the element will have a different number of flow lines, but the fixed points are independent of the representation. Therefor, it is immediate that the Cantor set underlying a component Z is independent of the revealing pair representing ϕ. Thus, we will also call this union a *component of support of* ϕ. The union of all the components of support of ϕ will be denoted $\overline{\text{Supp }(\phi)}$. Note that this is the topological closure of Supp ϕ. In other words, $\overline{\text{Supp }(\phi)} = \text{Supp }(\phi) \cup I(\phi)$.

Figure 2.2 is the flow graph of the element u given in Figure 2.1. Notice that u has 2 components of support.

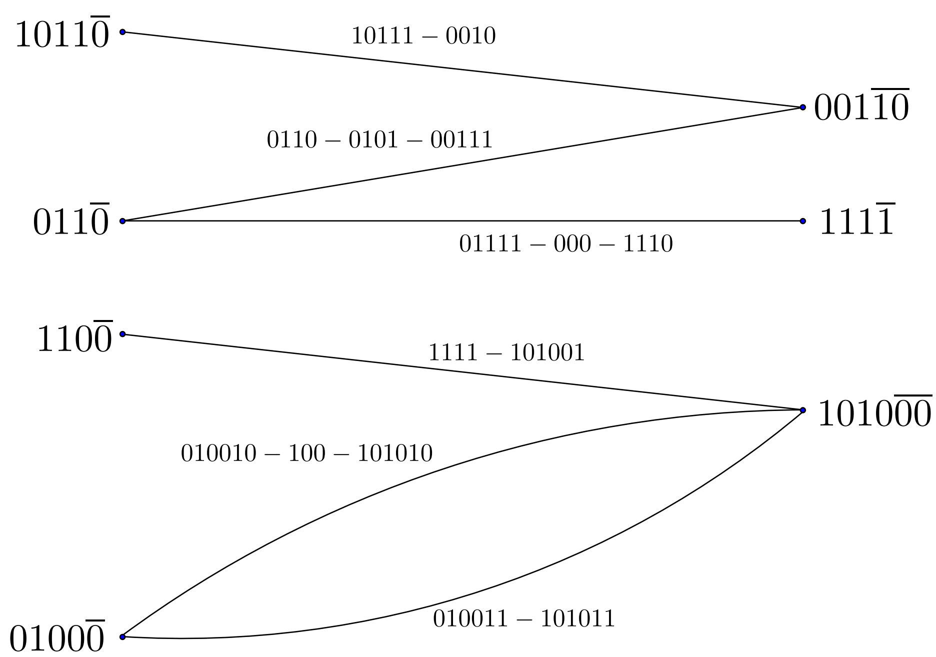


Figure 2.2: Flow Graph for the element u

2.5 Non-trivial Finite Periodic Orbits

In the previous section, and in most of the rest of this dissertation, results include the hypothesis that there are no non-trivial finite orbits in the action of the considered element. There are two types of non-trivial finite orbits. In the first, there is an entire neighborhood of the Cantor set in which every point has a non-trivial finite orbit. This neighborhood is not in the support of the element raised to an appropriate power. In the second type, every point near the point with a non-trivial finite orbit has an infinite orbit. When the element is raised to an appropriate power, the finite orbit points turn into important points. The element φ given in Figure 2.3 shows both scenarios.

Let $x = w \in \mathfrak{C}$. Then we note that $10x\varphi = 11x$ and $11x\varphi = 10x$. Hence,

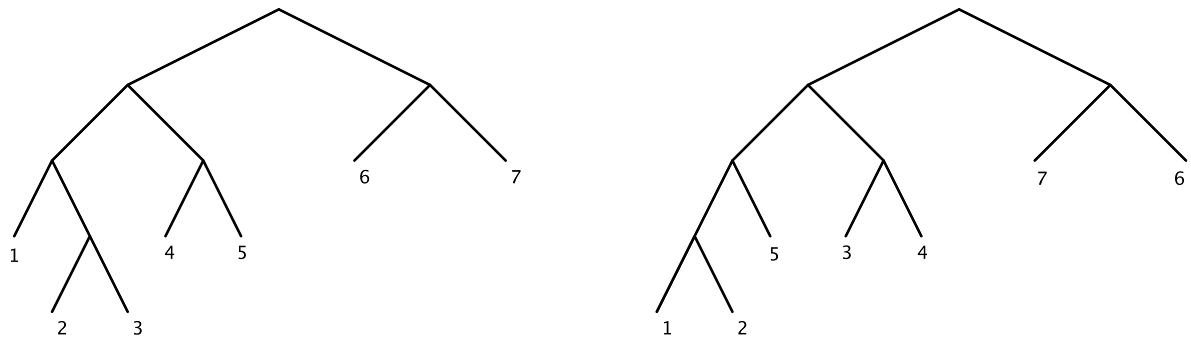


Figure 2.3: An element $\varphi \in V$ that has non-trivial periodic orbits.

every element of $\mathfrak{C}_1$ is on a periodic orbit of length 2. Next, note that $0011x\varphi = 010x, 010x\varphi = 011x$, and $011x = 001x$. In particular, $0011\overline{1}$ is on a periodic orbit of length 3, but no other point in $\mathfrak{C}_{0011}$ is.

As we wish to have no non-trivial periodic orbits, we need to raise φ to a multiple of both 2 and 3. Figure 2.4 is φ^6.

This is the general situation although the formal proof of this is technical. The following is a rewording of Proposition 10.1 in [9], the proof of which encompasses the entirety of Section 10.

Lemma 8. *(Brin, 04) Suppose $\mu \in V$. There is an integer ℓ such that μ^ℓ has no non-trivial periodic orbits.*

2.6 Wreath Products

Let A and T be groups. Set $B = \oplus_{t \in T} A$. Then, *the Wreath Product* of A and T is $A \wr T = B \rtimes T$ where the semi-direct product action of T on B is right multiplication on the index in the direct sum. We say T is the *top group*, A is the *bottom group*,

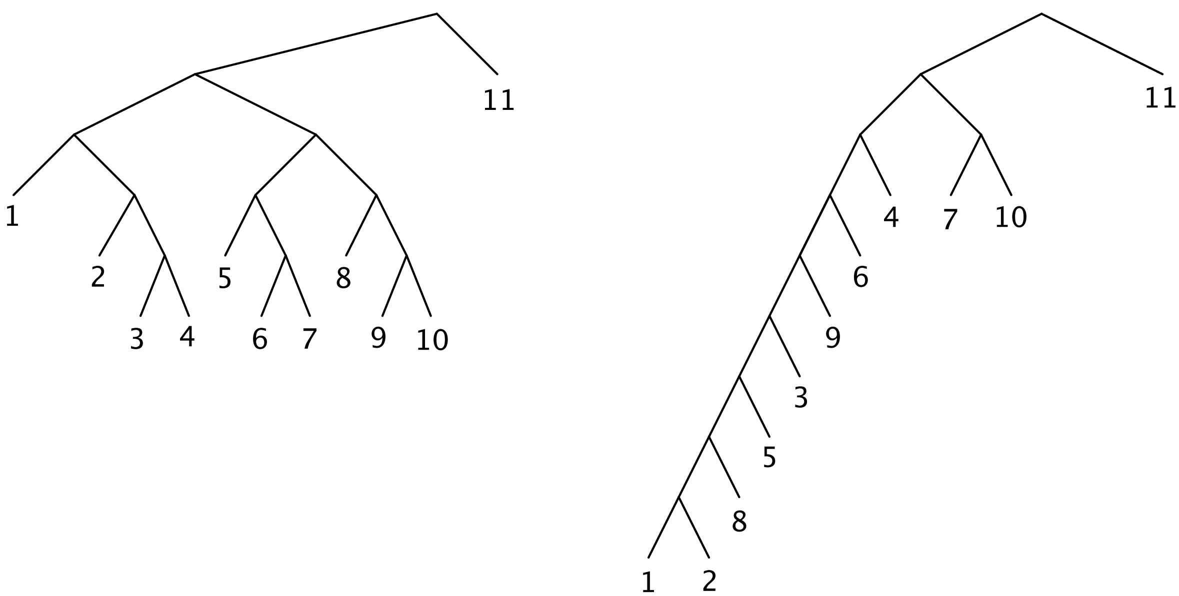


Figure 2.4: The element ϕ^6 has no non-trivial periodic orbits.

and B is called the *base group*.

For an example, let $b = (f_t)_{t\in T} \in B$. If we act on b by $s \in T$ we get a new element $b' \in B$ that is a shift. In particular, $b' = (f_{ts})_{t\in T}$. Said a different way, the element f_{t_0} of A that is in the entry of b indexed by t_0 is also the element in the entry of b' indexed by $t_0 s^{-1}$. The element of A that is in the coordinate of b' indexed by t_0 is the same element of A that is in the coordinate of b indexed by $t_0 s$.

Now, let $b = (f_t)_{t\in T}, b' = (g_t)_{t\in T} \in B$ and $r, s \in T$. Note that for every $t \in T$, we have f_t and g_t are elements of A. Then (b, r) and (b', s) are two elements of $A \wr T$. We can multiply these as follows:

$$(b, r) \cdot (b', s) = (\,(f_t)_{t\in T}, r) \cdot (\,(g_t)_{t\in T}, s) = (\,(f_t \cdot g_{tr})_{t\in T}, rs).$$

Chapter 3

$\mathbb{Z} \wr \mathbb{Z}^2$ does not embed into R. Thompson's group V

The key idea of the proof of the main result of this thesis is to suppose there is an embedding and build other embeddings from it. We will slowly improve the dynamics of the new injections until we have one which is too simple, deriving a contradiction. In order to do that, we need a few definitions in order to help us understand the dynamics.

Definition If $\alpha \in V, x \in \mathfrak{C}$ then we use $\mathcal{O}(\alpha, x)$ to denote $\{x\alpha^n | n \in \mathbb{Z}\}$, the orbit of x under α. For $\Theta \in V, X \subseteq \mathfrak{C}$, we say Θ *flees* X if $\forall x \in X$, $\mathcal{O}(\Theta, x) \cap X = \{x\}$. We say Θ *moves rapidly through* X, if $\forall x \in X$, $|\mathcal{O}(\Theta, x) \cap X| < \infty$.

We are mostly interested in the latter two definitions in order to rule out the existence of important points in certain regions.

Proposition 9. *If $\Theta \in V$ has no nontrivial finite orbits and moves rapidly through a conventional open set X, then $I(\Theta) \cap X = \emptyset$.*

Proof. We show the contrapositive. Suppose that $\Theta \in V$, X is a conventional open set, and $a \in I(\Theta) \cap X$. We wish to show that Θ does not moves rapidly through X. We will assume that a is an attracting fixed point, and that X is entirely contained in A, the attracting basin for Θ containing a. If a is a repelling fixed point, the same proof will work looking at the negative orbit, and if X is not entirely in the attracting (or repelling) basin, consider the intersection of X and the basin instead.

Fix $b \in X \setminus a$ and set $b_n = b\,\Theta^n$ for all integers n. As $b \neq a$ and is in A, $b_n \in A$ for all $n \geq 0$ by Lemma 6 and Corollary 7. Further, whenever $m \neq n$, $b_m \neq b_n$ since there are no nontrivial finite orbits and a is the unique fixed point in X. So, $|\mathcal{O}(\Theta, b) \cap X| = \infty$.

□

We will often times raise certain elements to powers. In general, this may introduce a new important point. The following proposition shows this cannot occur under controlled circumstances.

Proposition 10. *If $\Theta \in V$ flees a conventional open set X, and $n \in \mathbb{Z}$, then Θ^n also flees X. Similarly, if Θ moves rapidly through X, then so does Θ^n.*

Proof. It suffices to note that for all $x \in X$, $\{x\} \subseteq \mathcal{O}(\Theta^n, x) \subseteq \mathcal{O}(\Theta, x)$.
Thus, $1 \leq |\mathcal{O}(\Theta^n, x) \cap X| \leq |\mathcal{O}(\Theta, x) \cap X|$.

□

3.1 Some Relevant Facts

Say that α and β *have a common root* if there is a γ and integers n, m such that $\gamma^m = \alpha$ and $\gamma^n = \beta$. Here, γ is such a root. Brin and Squier showed in [11] that in F commuting one bump elements, i.e. elements with one component of support, either have disjoint support or have a common root. Not surprisingly, the situation is more complicated in V as seen in part (iii) of the following Lemma. This Lemma was first proved by Bleak and Salazar-Dìaz in [8] (as part of Lemma 2.5 and Lemma 2.6 in that paper) following the same underlying idea that Brin used. The proof is given since the techniques will be useful later on, and the author has streamline the proof of part (*ii*).

Lemma 11. *(Bleak, Salazar-Dìaz, 09) Suppose $g, h \in V$, each with no non-trivial periodic orbits. Suppose further, for (i) and (ii), that g and h commute. Then:*

i. $I(g) \cap I(h) = I(g) \cap \overline{\mathit{Supp}\ (h)} = I(h) \cap \overline{\mathit{Supp}\ (g)}$*;*

ii. If X and Y are components of support of g and h respectively, then $X = Y$ or $X \cap Y = \emptyset$;

iii. Suppose g and h have a common component of support X, and on X the actions of g and h commute. Then, there are non-trivial powers m and n such that $g^m = h^n$ over X.

Proof. For the first claim, since $\overline{\text{Supp}\ (h)} = \text{Supp}\ (h) \cup I(g)$, it is enough to show that if $x \in I(g) \cap \overline{\text{Supp}\ (h)}$ then $x \in I(h)$. If $x \notin I(h)$ then, for $n \in \mathbb{N}$, define $x_n = xh^n$. Then

$$x_n g^{h^n} = (xh^n)h^{-n}gh^n = xgh^n = xh^n = x_n.$$

So, $x_n \in I(g^{h^n})$. As g and h commute, $g^{h^n} = g$. Thus, every element in the orbit of x under h is an important point of g. This would imply that g has an infinite number of important points. As the number of important points of an element of V is always finite, we conclude $x \in I(h)$.

For the second, if $X \cap Y = \emptyset$ then the result is proved. So, we consider the case that $X \cap Y$ is not empty. As X and Y are each a finite union of basic open sets,

$X \cap Y$ is also a finite union of basic open sets. Thus, we can choose an $x \in X \cap Y$ that is not an important point of f or g.

Suppose there was a $m \in \mathbb{N}$ for which $xh^m \notin X$. Let $i = \lim xg^n$, so $i \in I(g)$. There are two cases to consider.

If $i \in \overline{\text{Supp }(h)}$ then $i \in I(h)$ as well by part (i). For sufficiently large s, $x_s := xh^s$ is close enough to i that $x_s h^m$ is still in the basin of attraction (with regard to g) containing i. Then, $xh^m g^s \notin X$, but $xg^s h^m \in X$.

Otherwise, $i \in \overline{\text{Supp }(h)}$. So, there is a sufficiently large integer n such that $xg^n \notin \overline{\text{Supp }(h)}$. Then $xg^n h^m = xg^n \in X \setminus Y$ and $xh^m g^n = xh^m \in Y \setminus X$.

In both case we have a contradiction as g and h commute. Thus $xh^m \in X$ for all m.

For the third claim, fix a particular revealing pair $P_g = (D_g, R_g, \sigma_g)$ for g along with a particular $p \in I(g) \cap I(h) \cap X$. As g and h both act as affine maps - each with slopes a power of 2 - close to p, there is neighborhood N_p and nonzero integers n, m such that $g^m = h^n$ on N_p. For each $q \in I(g) \cap X$ for which $g^m h^{-n}$ is trivial on a neighborhood of q, let N_q be such a neighborhood. Note that N_p is one such neighborhood. Let N be the union of all the N_q defined above.

If N is not a neighborhood of $I(g) \cap X$, then the connectivity of the component X of the flow graph of g implies there are $r, a \in I(g) \cap X$ with r a repeller and a an attractor. Further, exactly one of a and r is in N and the source-sink chain from the repelling basin B_r containing r to the attracting basin B_a containing a. In the case that r is not in N, there is a $y \in B_r \cap \text{Supp }(g^m h^{-n})$ and a positive integer k such that $xg^k \in N \cap B_a$. In the alternate case that a is not in N, there is a $y \in B_a \cap \text{Supp }(g^m h^{-n})$ and a negative integer k such that $xg^k \in N \cap B_r$. In either case, Lemma 5 shows that $(g^m h^{-n})^{g^k}$ has support where $g^m h^{-n}$ acts as the identity. But $(g^m h^{-n})^{g^k} = g^m h^{-n}$ as g and h commute. Hence N is a neighborhood of $I(g) \cap X$.

If $x \in \text{Supp}\,(g^m h^{-n}) \cap X$, then there is a sufficiently large $k \in \mathbb{N}$ such that xg^k is close enough to an attractor of g that $x \in N$. Again, this is a contradiction as $(g^m h^{-n})^{g^k}$ would have support in N. Thus, $g^m h^{-n}$ acts as the identity on X, hence $g^m = h^m$ when restricted to X. □

There is a collection of facts shown in [8] that we will use. The relevant part of that paper was focused on the non-embedding of $\mathbb{Z} * \mathbb{Z}^2$ into V. A construction called an (a, b, c)-commutator was used to great effect. The proofs of the facts in sections 4.2.2 and 4.2.3 of [8] go through to our setting with one change. We will now define an (a, b, c)-commutator and then prove one lemma of our own. With that done, we will then state and prove the results of [8] that we need.

Let Y be a non-empty set, closed under inverses and let $a, b, c, w \in Y^*$. We say that w is an *(a, b, c)-commutator* if there are integers $n > 0, x_i, y_i,$and z_i with $|x_i| + |y_i| \neq 0$ and $z_i \neq 0$ for all $0 \leq i \leq n$ such that in Y^*

$$w = [a^{x_1} b^{y_1}, [a^{x_2} b^{y_2}, \ldots [a^{x_{n-1}} b^{y_{n-1}}, [a^{x_n} b^{y_n}, c^{z_n}]^{z_{n-1}}]^{z_{n-2}} \ldots]^{z_1}].$$

We have one immediate fact.

Proposition 12. *(Bleak, Salazar-Dìaz, 09) Let $a, b, c \in Y^*$ for some nonempty set Y closed under inverses and suppose t is an (a, b, c)-commutator. If $0 \neq k \in \mathbb{Z}$ and w is an (a, b, t^k)-commutator, then w is also an (a, b, c)-commutator.*

The one result we need in order to adopt the facts in [8] mirrors the result that if $\mathbb{Z}^2 * \mathbb{Z} = \langle a, b, c | [a, b] \rangle$, t is an (a, b, c)-commutator, and k is a nonzero integer, then $\langle a, b, t^k \rangle$ factors as $\langle a, b \rangle * \langle t^k \rangle$. We will make use of the following.

Lemma 13. *Let $A \wr T = B \rtimes T$, with T torsion free. If $a \in T$ and $b \in B$ are both non-trivial then $[a, b]$ is a non-trivial element of the base.*

Proof. Say $a = (e_B, r)$ and $b = (\,(f_t)_{t\in T}, e_T)$ where e_B and e_T are the identity elements in B and T respectively. Then we compute

$$\begin{aligned}
[a,b] &= a^{-1}b^{-1}ab \\
&= (e, r^{-1})(\,(f_t^{-1})_{t\in T}, e)(e,r)(\,(f_t)_{t\in T}, e) \\
&= ((f_{tr^{-1}}^{-1})_{t\in T}, r^{-1})(e,r)(\,(f_t)_{t\in T}, e) \\
&= ((f_{tr^{-1}}^{-1})_{t\in T}, e)\cdot(\,(f_t)_{t\in T}, e) \\
&= ((f_{tr^{-1}}^{-1}\cdot f_t)_{t\in T}, e)
\end{aligned}$$

Thus, $[a,b]$ is in the base. It remains to show that it is non-trivial. As B is a direct sum, there are only a finite number of non-identity entries in b, say the ones indexed by $t_{\alpha_1},\ldots,t_{\alpha_N}\in T$ If for some ℓ, $t_{\alpha_\ell}r^{-1}\neq t_{\alpha_k}$ for any k, we are done as the entry indexed by t_ℓ in $[a,b]$ is $f_{\ell\cdot r^{-1}}^{-1}\cdot f_\ell = f_\ell \neq e_A$.

Otherwise, assume we have indexed in such a way that $t_{\alpha_1}r^{-1} = t_{\alpha_2}$. Note that $t_{\alpha_2}r^{-1} = t_{\alpha_1}r^{-2}$. Then, for some $1<\ell\leq N$, $t_{\alpha_1}r^{-\ell} = t_{\alpha_1}$. Thus, $r^{-\ell} = e_T$. This contradicts T being torsion free.

□

In particular, if a and b generate the top group $\mathbb{Z}^2$ and c is a non-trivial element of the base then the previous lemma tells us that any (a,b,c)-commutator is a non-trivial element of the base.

With that, we can now quote the results we need from sections 4.2.2 and 4.2.3 [8]. The proofs are given here for completeness and closely follow the original.

Lemma 14. *(Bleak, Salazar-Dìaz, 2009) Let $\mu,\nu,\rho\in V$. Further, suppose for parts (2) and (3) that $\mathbb{Z}\wr\mathbb{Z}^2 < V$.*

1. *If y is an important point of μ or ν, and ρ acts as the identity in some neighbor-*

hood U_y of y, then any (μ,ν,ρ)-commutator τ will act as the identity in some neighborhood V_y of y.

2. *Suppose ρ is a non-trivial element of the base, and μ,ν generate the top group $\mathbb{Z}^2$. If $I(\rho)\cap(I(\mu)\cup I(\nu))\neq\emptyset$, then there is a non-trivial element of the base τ such that, such that $I(\tau)\cap(I(\mu)\cup I(\nu))=\emptyset$. In particular, $\langle\mu,\nu,\tau\rangle\cong\mathbb{Z}\wr\mathbb{Z}^2$.*

3. *Suppose ρ is a non-trivial element of the base, and μ,ν generate the top group $\mathbb{Z}^2$. If $\overline{\mathit{Supp}\ (\rho)}\cap(I(\mu)\cup I(\nu))\neq\emptyset$, then there is a non-trivial element of the base τ such that $\overline{\mathit{Supp}\ (\tau)}\cap(I(\mu)\cup I(\nu))=\emptyset$. In particular, $\langle\mu,\nu,\tau\rangle\cong\mathbb{Z}\wr\mathbb{Z}^2$.*

Proof. (1) Let y and U_y be as in the hypothesis. Suppose that p and q are integers with not both equal to zero, and z a non-zero integer. Set $\tau=[\mu^p\nu^q,\rho^z]$. We will show that τ acts as the identitiy on a neighborhood N_y of y. The full claim follows from a straightforward induction.

Note that $\tau=(\rho^{-1})^{\mu^p\nu^q}\cdot\rho$, thus the support of τ is contained in Supp $(\rho)\cup$ (Supp $(\rho))\mu^p\nu^q$. Let M_y be a neighborhood of y disjoint from the action of ρ, say the intersection of U_y and the complement of the support of ρ, and set m to be the node in $\mathcal{T}$ whose cone set is M_y. Find a node n under m such that $\mu^p\nu^q$ acts affinely on the cone set B_n and $B_n\mu^{-p}\nu^{-q}\subset M_y$. As the support of ρ lies outside of M_y, the action of ρ on $\mathfrak{C}$ cannot throw the support of ρ into B_n. Thus, τ acts as the identity on B_n.

(2) Suppose there is an $x\in I(\rho)\cap(I(\mu)\cup I(\nu))$. Either $x\in I(\mu\nu)$, or $\mu\nu$ acts as the identity in a neighborhood of x. (The second case occurs when μ and ν are local inverses.) In either case, $\tau':=[\mu\nu,\rho]$ acts trivially in a neighborhood of x as τ' resolves as $\rho^{-1}\rho$ near x. Set $\tau=(\tau')^j$ where j is the minimal positive integer such that τ has no non-trivial finite orbits. Then $\langle\mu,\nu,\tau\rangle\cong\mathbb{Z}\wr\mathbb{Z}^2$ follows directly from Lemma 13.

If $I(\tau) \cap (I(\mu) \cup I(\nu)) \neq \emptyset$, repeat this process with τ taking the place of ρ in the previous paragraph. This process will terminate in a finite number of steps as the number of important points of μ and ν is finite, and (1) shows that each new iteration of τ will act as the identity in a neighborhood of each previously cleaned up important point.

(3) We may assume by (2) that $I(\rho) \cap (I(\mu) \cup I(\nu)) = \emptyset$. If Supp (ρ) contains none of the fixed points of μ and ν, there is nothing to be done. Otherwise, there is a $x \in$ Supp $(\rho) \cap (I(\mu) \cup I(\nu))$. Lemma 11 shows that x is not in the support of either μ or ν. We consider two cases based on how μ acts near $y = x\rho^{-1}$.

In the first, we assume y is disjoint from the support of μ. Define $\tau' = [\mu, \rho]$. Observe

$$x\tau' = x\mu^{-1}\rho^{-1}\mu\rho = x\rho^{-1}\mu\rho = y\mu\rho = y\rho = x.$$

Thus, x is fixed by τ'. In fact, the action of τ' fixes a small neighborhood of x by the fundamental theorem of calculus.

In the other case, y is in the support of μ. If y is also in the support of ν, then by Lemma 11, there are integers p and q such that $\mu^p\nu^q$ is trivial on their common component of support containing y. If y is not in the support of ν, let $p = 0$ and $q = 1$. Either way, $\tau' = [\mu^p\nu^q, \rho]$ fixes a neighborhood around x.

Now, regardless of how μ interacts with y, we have an (μ, ν, ρ)-commutator τ'. Set $\tau = (\tau')^j$ where j is the minimal positive integer such that τ has no non-trivial finite orbits. Again, Lemma 13 directly implies that $\langle \mu, \nu, \tau \rangle \cong \mathbb{Z} \wr \mathbb{Z}^2$.

Thus, if Supp $(\rho) \cap (I(\mu) \cup I(\nu)) \neq \emptyset$, the cardinality of Supp $(\tau) \cap (I(\mu) \cup I(\nu))$ is at least one smaller than $|$Supp $(\rho) \cap (I(\mu) \cup I(\nu))|$ as y has been removed and no new points have been added by (1). Applying the process described above may result with $I(\tau) \cap (I(\mu) \cup I(\nu)) \neq \emptyset$. However, again by (1), we will only have to apply (2) a finite number of times to eliminate common fixed points. We can then apply the process above to again reduce the size of $|\text{Supp }(\theta) \cap (I(\mu) \cup I(\nu))|$, where θ is the current generator of the base being considered. Thus, we find a τ as in the claim after a finite number of applications of the process in this part of the proof and of (2).

$\square$

3.2 Proof of Theorem

We now fix some notation that will be used throughout the remainder of this section. Consider $\mathbb{Z} \wr \mathbb{Z}^2$, with the top group generated by σ and τ, and the bottom group generated by ζ. Suppose that $\phi' : \mathbb{Z} \wr \mathbb{Z}^2 \to V$ is a group homomorphism. Set $s_0 = \sigma\phi'$ and $t_0 = \tau\phi'$. By Lemma 8, there is an integer ℓ such that s_0^ℓ and t_0^ℓ have no non-trivial periodic orbits. Define $s = (\sigma\phi')^\ell$ and $t = (\tau\phi')^\ell$. Set $\phi : \mathbb{Z} \wr \mathbb{Z}^2 \to V$ by $\zeta\phi = \zeta\phi'$, $\sigma\phi = s$, and $\tau\phi = t$. Note ϕ is also a homomorphism of $\mathbb{Z} \wr \mathbb{Z}^2$ into V. Further, if ϕ' is injective, so is ϕ.

Let $T = \langle t, s\rangle < V$ be the image under ϕ of the top group. Note that $T \cong \mathbb{Z}^2$ if ϕ is injective. From the discussion on flow graphs, T has a finite number of components of support, say $X_1, \ldots X_k$. Each component X_i is either a component of support of t or has empty intersection with the support of t. The same is true for s. Applying Lemma 11.3, for $i = 1, \ldots, k$, there are nontrivial words $u_i' = t^{r_i}s^{q_i}$ such that u_i' acts trivially on X_i. (If X_i is a component of support of t but not s, then one possibility is $u_i' = s$.) Let u_i be the least positive power of u_i' such that u_i contains no nontrivial finite orbits. Since u_i' acts trivially on X_i, it is immediate that u_i acts trivially on X_i as well.

Choose a non-trivial element γ of the base group. Apply Lemma 14.3 followed by Lemma 8 to γ to obtain a non-trivial (s, t, γ)-commutator $\gamma_0 \in V$ with no non-trivial finite orbits such that $\overline{\text{Supp }(\gamma_0)} \cap (I(s) \cup I(t)) = \emptyset$. Lemma 13 informs us that γ_0 is still in the base.

We will now build a series of non-trivial elements of the base group. Afterwards, we will investigate the properties of these new elements, eventually reaching a contradiction. The sequence will be constructed by removing the important points of γ_0 from the support of T one component of support at a time, starting with X_1. Recall that $1 \neq u_1 \in T$ acts as the identity on X_1. As $I(u_1) \subset I(s) \cup I(t)$, we have $I(u_1) \cap \overline{\text{Supp }(\gamma_0)} = \emptyset$. Thus, there is a power p_1 such that $\text{Supp }(u_1) \cap \overline{\text{Supp }(\gamma_0)} \cap \overline{\text{Supp }(\gamma_0) u_1^{p_1}} = \emptyset$. Set $w_1 = u_1^{p_1}$. Note that if $x \in \mathfrak{C}$ is in the support of both γ_0 and u_1, then w_1 moves x off the support of γ_0.

Define $\gamma_1' = [\gamma_0, w_1]$ and $\gamma_1 = (\gamma_1')^{K_1}$, where K_1 is the smallest positive integer such that γ_1 has no non-trivial finite orbits.

This process will be repeated. Once γ_{i-1} is defined, consider u_i. It is trivial over X_i and $I(u_i) \cap \overline{\text{Supp }(\gamma_{i-1})} = \emptyset$. Thus, there is a power p_i such that

$$\text{Supp }(u_i) \cap \overline{\text{Supp }(\gamma_{i-1})} \cap \overline{\text{Supp }(\gamma_{i-1}) u_i^{p_i}} = \emptyset.$$

Set $w_i = u_i^{p_i}$. Define $\gamma_i' = [\gamma_{i-1}, w_i]$ and $\gamma_i = (\gamma_i')^{K_i}$, where K_i is the smallest positive integer such that γ_i has no non-trivial finite orbits.

There are various scenarios when a point of $\mathfrak{C}$ is in the support of γ_i'. We will group the possibilities into three important classes.

Definition If $x \in \text{Supp } \gamma_i'$ we say x is i-*Type 1* if $x \in \text{Supp }(w_i) \cap \text{Supp }(\gamma_{i-1})$, i-*Type 2* if $x \in \text{Supp }(w_i) \setminus \text{Supp }(\gamma_{i-1})$ i-*Type 3* if $x \notin \text{Supp }(w_i)$.

When it is clear, we will talk about elements of Type 1 instead of i-Type 1 and similarly for the other types. Knowing which class an element of the support of γ_i' is in, along with knowing how γ_{i-1} and its inverse act on that element, will tell us what class γ_i' sends that element to along with a description of the new element.

Lemma 15. *Suppose $x_0 \in$ Supp γ_i' and, for $n \in \mathbb{Z}$ define $x_n = x_0 (\gamma_i')^n$. Then for fixed $m \in \mathbb{Z}$,*

1. *if x_m is of Type 1 and $x_m \gamma_{i-1}^{-1} \in$ Supp (w_i), then $x_{m+1} = x_m \gamma_{i-1}^{-1}$ and is of Type 1;*

2. *if x_m is of Type 1 and $x_m \gamma_{i-1}^{-1} \notin$ Supp (w_i), then $x_{m+1} = x_m w_i$ and is of Type 2;*

3. *if x_m is of Type 2 and $x_m w_i^{-1} \gamma_{i-1} \in$ Supp (w_i), then $x_{m+1} = x_m w_i^{-1} \gamma_{i-1} w_i$ and is of Type 2;*

4. *if x_m is of Type 2 and $x_m w_i^{-1} \gamma_{i-1} \notin$ Supp (w_i), then $x_{m+1} = x_m w_i^{-1} \gamma_{i-1} w_i = x_m w_i^{-1} \gamma_{i-1}$ and is of Type 3;*

5. *if x_m is of Type 3, then $x_{m+1} = x_m \gamma_{i-1}^{-1}$ and is of Type 1;*

6. *if x_m is of Type 2, then $x_m w_i^{-1} \in$ Supp (γ_{i-1}).*

Additionally, this list is a complete list of possibilities.

Proof. The proofs for all five parts are similar and involve tracking the elements. Since $\gamma_i' = \gamma_{i-1}^{-1} w_i^{-1} \gamma_{i-1} w_i$, we will ease the tracking by defining $a = x_m \gamma_{i-1}^{-1}, b = a w_i^{-1}$, and $c = b\gamma_{i-1}$. We note that $x_{m+1} = c w_i$.

For part 1, the assumption is that $a \in$ Supp (w_i) and, as $x_m \in$ Supp (γ_{i-1}), so is a. Hence, $b \notin$ Supp (γ_{i-1}) and so $c = b$. So, $x_{m+1} = a$.

For part 2, $a \notin$ Supp (w_i), so $b = a$ and thus $c = x_m$. So, $x_{m+1} = x_m w_i$.

For part 6, assume x_m is of Type 2. If $x_m w_i^{-1} \notin \text{Supp}\,(\gamma_{i-1})$ then we have $b = c$, and thus $a = d$. Since $x_m = a$ always in Type 2, this would imply that $x_m = x_{m+1}$, contradicting $x_m \in \text{Supp}\,(\gamma_i')$.

For part 3 and 4, $x_m = a$. As stated above, $b \in \text{Supp}\,(\gamma_{i-1})$, so $c \neq b$. Thus, $x_{m+1} = x_m w_i^{-1} \gamma_{i-1} w_i$. In part 4, the last w_i acts trivially, while it doesn't in part 3.

For part 5, we first consider the case where $a \notin \text{Supp}\,(w_i)$. Then $b = a$ and $c = x_m$. Thus $x_{m+1} = x_m$ as x_m is fixed by w_i. However, x_m is in the support of γ_i'. Thus, we know that $a \in \text{Supp}\,(w_i) \cap \text{Supp}\,(\gamma_{i-1})$. Thus, b is not in the support of γ_{i-1}, hence $c = b$ and $x_{m+1} = a$.

$\square$

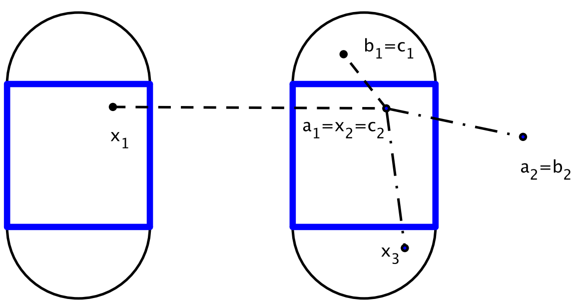


Figure 3.1: A picture of the proof of parts 1 and 2 of Lemma 15. The ovals are components of support of w_i, the squares are supports of γ_{i-1} inside the supports of w_i. In this example, both x_1 and x_2 are of i-Type 1 and x_3 is of i-Type 2.

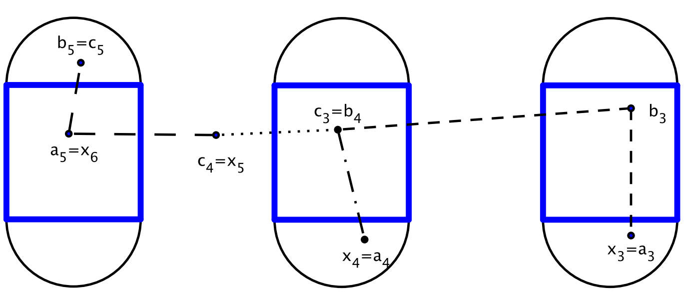


Figure 3.2: A picture of the proof of parts 3, 4 and 5 of Lemma 15. In this example, x_3 and x_4 are of i-Type 2, while x_5 is of i-Type 3 and x_6 is of i-Type 1.

Figure 3.1 and Figure 3.2 illustrate the proof of Lemma 15. The preceding lemma is quite powerful. For example, it enables us to show that our aim to construct each of the elements γ_i to have no important points in the corresponding X_i is achieved.

Proposition 16. *If $x \in X_i$, $n \in \mathbb{N}$, and $y = x(\gamma_i')^n \in X_i$, then $y = x$.*

Proof. Let $x_n = x(\gamma_i')^n$. As $x = x_0 \in X_i$, and w_i acts trivially on X_i, x_0 is of Type 3. Hence, by Lemma 15, we have $x_1 = x\gamma_{i-1}^{-1}$ and is of Type 1. By that same Lemma, we also know that there is an m such that $x_r = x\gamma_{i-1}^{-r}$ and is of Type 1 for all $1 \leq r \leq m$ but x_{m+1} is of Type 2. The other possibility is that x_r is of Type 1 for all $r > 0$, but by hypothesis the positive orbit eventually returns to the support of w_i and hence must pass through a Type 2 element on its way to a Type 3 element. Then, $x_{m+1} = x_m w_i$. Applying Lemma 15.3 $m-1$ more times finds $x_{2m} = x_1 w_i$. A single application of Lemma 15.4 then show $x_{2m+1} = x_{2m} w_i^{-1}\gamma_{i-1} = x_1 w_i w_i^{-1}\gamma_{i-1} = x_1\gamma_{i-1} = x_0$.

Thus, $|\mathcal{O}(x, \gamma_i')| < \infty$. Also, $x_r \in$ Supp (w_i) for $1 \leq r \leq m$, and in particular is not in X_i. Thus, $\mathcal{O}(x, \gamma_i') \cap X_i = \{x\}$.

□

Proposition 17. *For any i, γ_i flees X_i. In particular, $I(\gamma_i) \cap X_i = \emptyset$.*

Proof. Suppose $x \in$ Supp $(\gamma_i') \cap X_i$. Proposition 16 shows that if $x \in X_i$ and there is a positive integer n such that $x(\gamma_i')^n \in X_i$, then the orbit of x under γ_i' is periodic. Hence, $\mathcal{O}(x, \gamma_i) = \{x\}$.

It cannot be the case that there is no positive integer such that $x(\gamma_i')^n \in X_i$, but there is a negative integer n such that $y = x(\gamma_i')^n \in X_i$, as considering y instead of x with the above argument shows that $y = x$.

We are left to conclude that $\mathcal{O}(x, \gamma_i') \cap X_i = \{x\}$. Then $\mathcal{O}(x, \gamma_i) \cap X_i = \{x\}$. The second sentence of the statement of the corollary now follows from Proposition 10.

□

Recall that our strategy is to recursively build the γ_i's. Thus, it is not sufficient to simply ensure that there are no important points of γ_i in X_i. We also need to ensure there are no important points of γ_i in X_j for all $1 \leq j \leq i$. Informally, we cleaned up these X_j's previously be eliminating the important points, and we wish to not mess than up again while cleaning X_i. The following Lemma shows that the process described above does not add any important points into the previously fixed components of support.

Lemma 18. *If γ_{i-1} moves rapidly though X_j for all $j < i$, then γ_i also moves rapidly though X_j for all $j < i$. Additionally, for no $x \in \mathfrak{C}$ does $\mathcal{O}(x, \gamma_i)$ have more than one element of i-Type 3.*

Proof. Fix a $j < i$ and an $x \in X_j \cap \mathrm{Supp}\ \gamma_i'$. Recall that $\gamma_i' = \gamma_{i-1}^{-1}\gamma_{i-1}^{w_i}$. If w_i is trivial over X_j then γ_i' is as well and the result is immediate. Otherwise, the support of (γ_{i-1}^{-1}) is disjoint from the support of $\gamma_{i-1}^{w_i}$ inside X_j and x is in the support of exactly one of (γ_{i-1}^{-1}) and $\gamma_{i-1}^{w_i}$. We consider cases, observing that the above shows that x is not of Type 3.

1. The element x, and every point in $\mathcal{O}(x, \gamma_i')$, is of i-Type 1.

   Define $x_n := x\gamma_{i-1}^{-n}$. Then, by Lemma 15.1, we see that $x_n = x(\gamma_i')^n$ for all $n \in \mathbb{Z}$. Thus, $\mathcal{O}(x, \gamma_{i-1}) = \mathcal{O}(x, \gamma_i')$. As $\mathcal{O}(x, \gamma_i) \subseteq \mathcal{O}(x, \gamma_i')$, we have $|\mathcal{O}(x, \gamma_i) \cap X_j| \leq |\mathcal{O}(x, \gamma_i') \cap X_j| = |\mathcal{O}(x, \gamma_{i-1}) \cap X_j| < \infty$.

2. The element x, and every point in $\mathcal{O}(x, \gamma_i')$, is of i-Type 2.

   Define $x_n := xw_i^{-1}\gamma_{i-1}^{n}w_i$. Then, by Lemma 15.1, we see that $x_n = x(\gamma_i')^n$ for all $n \in \mathbb{Z}$. Let y denote the point xw_i^{-1} and for all integers n let $y_n = y\gamma_{i-1}^{-n}$. Then we note that y satisfies the hypotheses of Case 1. Thus, $\mathcal{O}(y, \gamma_i') = \mathcal{O}(y, \gamma_{i-1})$.

We also note that $x_n = y_n w_i$, in particular x_n and y_n share a component of support. Hence, $|\mathcal{O}(x, \gamma_i') \cap X_j| = |\mathcal{O}(y, \gamma_{i-1}) \cap X_j|$ which is finite by assumption.

3. Every point in $\mathcal{O}(x, \gamma_i')$, is of i-Type 1 or of i-Type 2, with both types occurring.

   Suppose at first that the element x is of Type 1. Informally, in this case, iterations of x by γ_i' follows the orbit of x under γ_{i-1} in the reverse order until the orbit of γ_{i-1} moves into a component of support that is fixed by w_i. Instead of following into this component, future iterations nearly turn around and retrace its steps, except the sequence is now shifted by the action of w_i.

   Formally, define $x_n := x(\gamma_i')^n$. Let N be the index such that x_N is of Type 1, but x_{N+1} is of Type 2. Lemma 15 guarantees both the existence and the uniqueness of N. If we call $y = X_N$ and $y_p = y\gamma_{i-1}^p$, then we have for $p \leq N$ that $x_p = y_{N-p}$. Continuing to use Lemma 15, we see that $x_{N+1} = y_0 w_i$ and, in general for $p > N$, $x_p = y_{p-N-1} w_i$. In particular, if a is a lesser integer than b and $a + b = 2N + 1$, then x_a and x_b are related by $x_b = x_a w_i$ and thus are in the same component of support. Further, $x_a = y_{a-N} \in \mathcal{O}(x, \gamma_{i-1})$. In X_j there are at most two elements of the orbit of x under γ_i' for each element of $\mathcal{O}(x, \gamma_{i-1}) \cap X_j$. Thus, $|\mathcal{O}(x, \gamma_i) \cap X_j| \leq |\mathcal{O}(x, \gamma_i') \cap X_j| \leq 2|\mathcal{O}(x, \gamma_{i-1}) \cap X_j| < \infty$.

   If x is instead an element of Type 2, then the above can be easily modified, or one can consider $z = xw_i^{-1}$ in the preceding paragraph.

4. There is an element in $\mathcal{O}(x, \gamma_i')$ that is of Type 3.

   Let y be an element of $\mathcal{O}(x, \gamma_i')$ that is of Type 3 and define $y_n = y(\gamma_i')^n$ for integers n. Note $y = y_0$. By definition, $\mathcal{O}(y, \gamma_i') = \mathcal{O}(x, \gamma_i')$. By Lemma 15.5, $y_1 = y\gamma_{i-1}^{-1}$ is of Type 1. This is equivalent to writing $y_1 \gamma_{i-1} = y_0$. By Lemma 15.4, y_{-1} is of Type 2 and $y_0 = y_{-1} w_i^{-1} \gamma_{i-1}$. We now have two alternative ways of expressing y_0, and setting them equal to each other obtains $y_1\gamma_{i-1} = y_{-1}w_i^{-1}\gamma_{i-1}$. Thus, $y_1 = y_{-1}w_i^{-1}$.

Now suppose, for $N > 0$, that y_N is of Type 1, y_{-N} is of Type 2, and $y_N = y_{-N}w_i^{-1}$. Again referring to Lemma 15, we see that $y_{-(N+1)}$ is either of Type 1 or of Type 2. If $y_{-(N+1)}$ is of Type 2, then $y_{-(N+1)}w_i^{-1}\gamma_{i-1} \in \text{Supp }(w_i)$. We also can write

$$\begin{aligned} y_{-N} &= y_{-(N+1)}w_i^{-1}\gamma_{i-1}w_i \\ y_N w_i &= y_{-(N+1)}w_i^{-1}\gamma_{i-1}w_i \\ y_N &= y_{-(N+1)}w_i^{-1}\gamma_{i-1} \\ y_N\gamma_{i-1}^{-1} &= y_{-(N+1)}w_i^{-1} \\ y_{N+1} &= y_{-(N+1)}w_i^{-1} \end{aligned}$$

In particular, if y_{-N} is of Type 2 for all $N > 0$, then y_N is of Type 1 for all $N > 0$. Thus $\{y(\gamma_i')^n : n > 0\} \subset \mathcal{O}(y, \gamma_{i-1})$ and $\{y(\gamma_i')^n : n < 0\} \subset \mathcal{O}(y, \gamma_{i-1})w_i^{-1}$. Note that $|\mathcal{O}(y, \gamma_{i-1})w_i^{-1} \cap X_j| = |\mathcal{O}(y, \gamma_{i-1}) \cap X_j| = |\mathcal{O}(x, \gamma_{i-1}) \cap X_j|$. In this subcase, $\mathcal{O}(x, \gamma_i)$ only has one element not in Supp (w_i). As $\mathcal{O}(y, \gamma_i') = \mathcal{O}(x, \gamma_i')$ we have $|\mathcal{O}(x, \gamma_i') \cap X_j| \leq 2|\mathcal{O}(x, \gamma_{i-1}) \cap X_j| + 1 < \infty$.

On the other hand, $y_{-(N+1)}$ might be of Type 1. Then, Lemma 15.2 informs us that $y_{-(N+1)}w_i = y_{-N}$. Thus $y_{-(N+1)} = y_{-N}w_i^{-1} = y_N$. Thus, $|\mathcal{O}(x, \gamma_i')| < \infty$ and thus $|\mathcal{O}(x, \gamma_i)| = \{x\}$.

□

We are now able to prove the main theorem.

Theorem 19. *$\mathbb{Z} \wr \mathbb{Z}^2$ does not inject into V.*

Proof. Consider again the homomorphism ϕ' of $\mathbb{Z} \wr \mathbb{Z}^2$ into V. Assume, for the sake of contradiction, that ϕ' is injective, hence ϕ is as well. Now consider γ_0, the non-trivial element of the base group with $\overline{\text{Supp }(\gamma_0)}$ disjoint from the important points of s and t as discussed at the beginning of this section. By Proposition 17, we see that γ_1 has no important points in X_1. Recursively, if γ_i has no important points in X_j for $j \leq i$, then - by another application of Proposition 17 - we see that γ_{i+1} has no important points in X_{i+1} and Lemma 18 further implies that γ_{i+1} has no important points in X_j for $j \leq i$.

In particular, γ_k has no important points in any of the components of support of w_k. By Lemma 18, if $x \in \mathfrak{C}$, then $\mathcal{O}(x, \gamma_k)$ has at most one point y not in Supp (w_k). Taking the cone set of any node above y that is not in the support of w_k and applying Proposition 9 shows that γ_k has no important points outside the components of support of w_k and hence $I(\gamma_k) = \emptyset$. This implies that γ_k is of finite order. As $\mathbb{Z} \wr \mathbb{Z}^2$ is torsion free, and γ_k is in the image of ϕ, we see that γ_k must be trivial.

However, each γ_i' is the commutator of a non-trivial element of the base group and a non-trivial element of the torsion free top group $\mathbb{Z}^2$. Thus, by Lemma 13 each γ_i' is non-trivial and of infinite order, in particular γ_k is non-trivial. This is a contradiction. We conclude there is no such injection ϕ'.

$\square$

Chapter 4

Generalizations of V

There are various ways to generalize Thompson's group V. Two will be discussed in this section. In both subsections, the class of groups will be defined and results similar to the above will be discussed.

4.1 Higman-Thompson Groups

The Higman-Thompson groups $G_{n,r}$ were first described in 1970 by Higman in [16]. These were an infinite family of finitely presented groups that are simple (when n is even) or contain an index 2 simple group (when n is odd). Fix integers $r \geq 1$ and $n \geq 2$. We will now describe the group $G_{n,r}$.

Define $T = \{\tau_1, \ldots, \tau_r\}$ and $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ to be disjoint sets of cardinality r and n, respectively. Set $\Omega = T\Sigma^\omega$, i.e., the set of all infinite sequences of the form $w = \tau_i \alpha_{i_1}, \alpha_{i_2}, \ldots$ with $\tau_i \in T$ and $\alpha_{i_j} \in \Sigma$ for all $j \geq 1$. The set of finite sequences $T\Sigma^*$ is defined analogously. Note that $T\Sigma^*$ is precisely the set of prefixes for Ω.

We define a *barrier* as a finite subset B of $T\Sigma^*$ such that each $w \in \Omega$ has exactly one element of B as a prefix. As an example, $\{\tau_1\alpha_1\alpha_1, \tau_1\alpha_1\alpha_2, \tau_1\alpha_2, \tau_2\}$ is a barrier when $n = r = 2$ but neither $\{\tau_1\alpha_1, \tau_1\alpha_1\alpha_2, \tau_1\alpha_2, \tau_2\}$ nor $\{\tau_1\alpha_1, \tau_1\alpha_2, \tau_2\alpha_1\}$ is a barrier in this case.

A *prefix replacement* is a triple $f = (D, R, \sigma)$ where D and R are barriers and σ is a bijection between D and R. The prefix replacement f defines a *prefix replacement permutation* of Ω as follows: for $w \in \Omega$, wf is the string w but with the unique prefix $p \in D$ of w replaced with the prefix $p\sigma \in R$. One should observe that different prefix replacements can induce the same prefix replacement permutation.

Consider the set $G_{n,r}$ of all prefix replacement permutations of Ω and the operation of composition. That the composition of prefix replacement permutations is a prefix replacement permutation is most easily seen by considering expansions. The *expansion* of a barrier B is a new barrier B' obtained as follows: let $p \in B$. Then B' contains all of B except p and including the n elements of the form $p\sigma_j$. Building on this, the *expansion* of a prefix replacement $f = (D, R, \sigma)$ is a new triple $f' = (D', R', \sigma')$ where D' is the expansion of D at $p \in D$, R' is the expansion of R at $p\sigma$. The new function σ' acts the same as σ on all elements of $B' \setminus B$ and maps the new elements by $p\alpha_j\sigma' = p\sigma\alpha_j$. It is immediate that f and f' induce the same map on Ω. Suppose $f_1, f_2 \in G_{n,r}$ with f_i induced by (D_i, R_i, α_i) for $i = 1, 2$. Then we can repeatedly expand both triples such that f_i is induced by (D_i', R_i', α_i') for $i = 1, 2$ and $R_1' = D_2'$. Thus f_1f_2 is induced by $(D_1', R_2', \sigma_1'\sigma_2')$.

It is immediate that $G_{n,r}$ is a group. The identity element is realized by the triple $(I, I, 1)$ where $I = T$ and 1 is the identity map. The inverse of an element induced by $f = (D, R, \sigma)$ is induced by (R, D, σ^{-1}). Composition of functions is well known to be associative.

The above description is not the most common one for Higman-Thompson groups, however it is very similar to the approach used by Holt and Röver in [18] and based off the idea by Thompson in [24] that we can think of these groups as prefix replacements.

In the original description, these groups are automorphism groups of algebras. The groups can also be defined almost exactly as in the description of V above, only with D and R each being a forest of r trees rather than just one tree each, and each tree in the forest being an n-ary tree rather than the binary tree. In any description, it is immediate that $G_{2,1}$ is isomorphic to V.

The following result is well known, and is first shown as part of the proof of Theorem 7.3 in [16],

Lemma 20. *(Higman, 74) Let $r \geq 1$ and $n \geq 2$ be integers. Then $G_{n,r}$ embeds into $G_{2,1}$.*

This leads to a natural extension of Theorem 19.

Corollary 21. *$\mathbb{Z} \wr \mathbb{Z}^2$ does not inject into $G_{n,r}$ for any pair of integers $r \geq 1$ and $n \geq 2$.*

Proof. Suppose there were integers $r \geq 1$ and $n \geq 2$ and an injective group homomorphism $\phi : \mathbb{Z} \wr \mathbb{Z}^2 \to G_{n,r}$. By Lemma 20, there is an injective group homomorphism $\psi : G_{n,r} \to G_{2,1} = V$. Thus $\phi \circ \psi$ is an injective group homomorphism from $\mathbb{Z} \wr \mathbb{Z}^2$ to V, contradicting Theorem 19.

$\square$

4.2 Brin's Groups nV

In this subsection, we discuss a different generalization of Thompson's group V. The groups nV, defined for positive integers n, also act on the Cantor set, although we will think of them acting on n dimensional Cantor dust $\mathfrak{C}^n$ endowed with the product topology. In this sense, we think of nV as a higher dimensional version of $V = 1V$.

Informally, we can interpret Theorem 19 to say that a rank two free abelian group joined to $\mathbb{Z}$ with a wreath product does not embed into $1V$. In the following, we will carefully define the groups nV. We will then show two positive results in the same flavor as Theorem 19, namely that a rank n free abelian group joined with $\mathbb{Z}$ by a wreath product does embed into nV, as does a rank n free abelian group joined with $\mathbb{Z}$ by a free product.

4.2.1 Description of nV

Recall that the cone set of $b \in \{0,1\}^*$ is $B_b = \{x \in \mathfrak{C} | x = by$ for some $y \in \mathfrak{C}\}$, the set of all elements of $\mathfrak{C}$ with b as a prefix, and all of the sets of this form defines a basis for $\mathfrak{C}$. An element of V can be described by two collections, D and R, of cone sets of $\mathfrak{C}$ that each partition $\mathfrak{C}$ and a bijection between the two collections. The map then replaces the prefix of an element of the Cantor set represented in D by the associated prefix in R.

We will use the following convention for the rest of this thesis: if we have a finite list of strings $b_1, \ldots, b_n \in \{0,1\}^*$, then we will use $B_1, \ldots, B_n$ to refer to their respective cone sets. In $\mathfrak{C}^n$, a subset R is an *n-rectangle* if there are strings $p_1, \ldots, p_n \in \{0,1\}^*$ such that $R = (P_1, \ldots, P_n)$.

Suppose $D = (P_1, \ldots, P_n)$ and $R = (Q_1, \ldots, Q_n)$ are two n-rectangles. The *n-rectangle map* $\tau_{D,R} : D \to R$ maps z to z' where the i-th coordinate of z is $p_i z_i$ and the i-th coordinate of z' is $q_i z_i$. In other words, $\tau_{D,R}$ affinely maps the n-rectangle D to the n-rectangle R by prefix replacement in each coo.

A *pattern* is a partition of $\mathfrak{C}^n$ into a finite collection of n-rectangles. An element f of nV is a homeomorphism of $\mathfrak{C}^n$ that can be represented by a domain pattern $\mathcal{D}$, a range pattern $\mathcal{R}$, and a bijection β between the n-rectangles in $\mathcal{D}$ and the n-rectangles

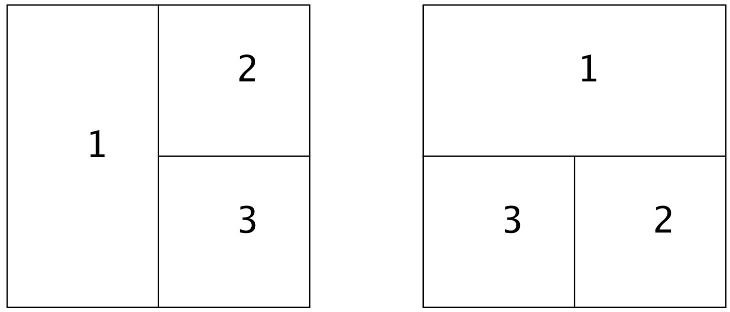


Figure 4.1: The element $\phi \in 2V$, represented by 2-rectangles.

in $\mathcal{R}$ such that f restricted to $D_i \in \mathcal{D}$ acts as τ_{D_i,R_i}. One should note that different pairs of patterns can represent the same element of nV.

Figure 4.1 is a visual representation of the element $\phi \in 2V$ with the following three 2-rectangles in each its domain and range pattern:

$$
\begin{aligned}
D_1 &= (0, \emptyset) \qquad & R_1 &= (\emptyset, 0)\\
D_2 &= (1, 0) \qquad & R_2 &= (1, 1)\\
D_3 &= (1, 1) \qquad & R_3 &= (0, 1)
\end{aligned}
$$

The horizontal dimension in the figure corresponds to the first coordinate in the algebraic description and the vertical dimension corresponds to the second coordinate. The points on each axis are in binary with 0 drawn to the left and top and 1 on the right and bottom. For example, note that $(0110x, 0101y)\phi = (110x, 00101y)$ and $(11001x, 10110y)\phi = (01001x, 10110y)$.

4.2.2 Baker's Map

One might try to show some sort of generalization of Theorem 19 for the groups nV by following a similar strategy to the previous proof, however, there is an immediate

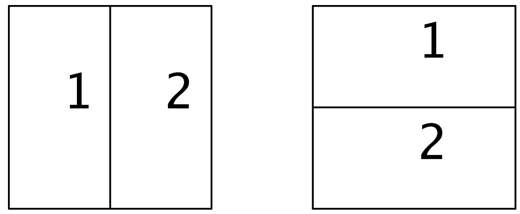


Figure 4.2: The bakers map in $2V$.

obstacle. The proof to Theorem 19 depended heavily on the dynamics of the action of V on $\mathfrak{C}$. The dynamics were completely describable in $1V$, but can be much more challenging to understand in nV for $n \geq 2$.

For example, Figure 4.2 shows a $2V$ version of the baker's map. It is called this because the action on $\mathfrak{C}^2$ should remind the reader of kneading dough. We will focus this discussion on $2V$. Hopefully it will be evident how to make functions as least as complicated dynamically in higher dimensions.

The following elegant argument was given in [9], where it is attributed to Dennis Pixton. Just for this section, think of a point in $\mathfrak{C} \times \mathfrak{C}$ as a doubly infinite sequence in the following way. The first coordinate will be written from left to right, and the second coordinate from right to left. Placing the two sequences next to each other, separated by a "*binary point*", allows us to view the the point as a function from $\mathbb{Z}$ to $\{0, 1\}$. If we write x_i for the image of i and place the binary point between x_{-1} and x_0, then the first coordinate of the point is represented by the sequence $(x_i)_{i \geq 0}$ and the second coordinate is represented by the sequence $(x_{-i})_{i>0}$. The differing conventions of the sequences, with the first coordinate have an index starting at 0 and the second at 1, is necessary to set up the next statement, which is Lemma 8.1 in [9].

Lemma 22. *(Brin, 04) The baker's map corresponds to shifting a doubly infinite sequence from* $\{0, 1\}$ *one position. Specifically, if* b *is the baker's map, and* $x : \mathbb{Z} \to$

$\{0,1\}$ *is a sequence representing an element of* $\mathfrak{C} \times \mathfrak{C}$*, then* $(b(x))_i = x_{i+1}$.

Proof. There are two cases to consider. In the first, $x_0 = 0$. Thus, x is located in the left half of the domain rectangle in Figure 4.2. Thus, $b(x)$ has a zero removed in the first location of the first coordinate, and 0 added in the first location of the second coordinate. This is equivalent to moving the binary point to the left one place. The second case, when $x_0 = 1$, is similar.

□

This has a collection of consequences that we will be interested in, the first is explicitly mentioned in [9].

Corollary 23. *1. There is no bound on the size of the finite orbits of the baker's map.*

2. The set of points in $\mathfrak{C} \times \mathfrak{C}$ *with finite orbits under the baker's map is dense in* $\mathfrak{C} \times \mathfrak{C}$*.*

3. The set of points in $\mathfrak{C} \times \mathfrak{C}$ *with infinite orbits under the baker's map is dense in* $\mathfrak{C} \times \mathfrak{C}$*.*

Proof. For the first, note that any periodic function $x : \mathbb{Z} \to \mathfrak{C} \times \mathfrak{C}$ with period p lies in a finite orbit of the baker's map with period p by Lemma 22.

For the second, let $S = (s_1, s_2)$ be any rectangle. It suffices to show that there is a point of finite order in S. Consider the point given by the periodic function s whose image is the repeated string $s_1 s_2^{-1}$ (where s_2^{-1} is the string s_2 in reverse order) and for which the binary point is between the last character of s_1 and the last character of s_2 (hence the first of s_2^{-1}). Then $s \in S$ and has a finite orbit under the baker's map.

For the last, again let $S = (s_1, s_2)$ be any rectangle. It suffices to show that there is a point of infinite order in S. Consider a non-periodic function t whose image has

s_1 on the right of the binary point and s_2 in reverse on the left. Then $t \in S$ and, as t is non-periodic, the image of t under the baker's map has infinite order.

□

Notice the difference between Lemma 8 and Corollary 23.1. It was this observation that the baker's map has non-trivial entropy while every element in V has trivial entropy that encompassed the proof that V is not isomorphic to $2V$. This also seems to make difficult the task of properly defining the analogue to important points.

The remainder of this thesis will show the existence of some embeddings into nV.

4.2.3 Free Products, the Ping Pong Lemma, and an Embedding Result

The free product of two groups is a well known algebraic construction. Given two subgroups $H_1, H_2 < H$, a basic question is whether the group $\langle H_1, H_2 \rangle$ generated by H_1 and H_2 is the free product $H_1 * H_2$. One set of sufficient conditions for this decomposition to exist is given by the ping-pong lemma, a technique attributed to Felix Klein in the late 1800's. It is known that the hypotheses are not necessary; there are free products that do not arise from a ping-pong. The following version is Item 24 of Chapter 2 in [14].

Lemma 24. *Let H be a group acting on a set X, and let H_1, H_2 be two subgroups of H with H_1 containing at least 3 elements and H_2 containing at least 2. Suppose that there exist two non-empty subsets X_1 and X_2 of X with X_2 not included in X_1 such that*

$$X_2 h_1 \subseteq X_1 \text{ for all } h_1 \in H_1,$$

$$X_1 h_2 \subseteq X_2 \text{ for all } h_2 \in H_2.$$

Then, $\langle H_1, H_2 \rangle$ *is isomorphic to* $H_1 * H_2$*.*

We can now prove our result about free products and nV.

Theorem 25. *For all* $n \in \mathbb{N}$*,* $\mathbb{Z}^n * \mathbb{Z}$ *embeds into* nV*.*

Proof. Fix $n \in \mathbb{N}$. We will construct a subset $A = \{a_1, \ldots, a_n\}$ of nV such that each a_k is of infinite order and a_k commutes with a_j if $j \neq k$. Further, we will show that if $a_1^{r_1} \ldots a_n^{r_n} = 1$ then $r_1 = \cdots = r_n = 0$. Thus $\langle A \rangle \cong \mathbb{Z}^n$. We will then construct an element b such that $\langle b \rangle \cong \mathbb{Z}$. Lastly, we will apply Lemma 24 to conclude $\langle A, b \rangle \cong \mathbb{Z}^n * \mathbb{Z}$.

For $k \in \{1, \ldots, n\}$, define an element $a_k \in nV$ as follows. The set $\mathfrak{C}^n$ is partitioned into 4 n-rectangles for both the domain and the range partition. The bijection is given by the subscripts and all non-trivial prefixes occur in the k-th coordinate. The domain n-rectangles (denoted D_i) and the range n-rectangles (denoted R_i) are:

$$\begin{aligned}
D_1 &= (\emptyset, \ldots, \emptyset, 000, \emptyset, \ldots, \emptyset) & R_1 &= (\emptyset, \ldots, \emptyset, 00, \emptyset, \ldots, \emptyset) \\
D_2 &= (\emptyset, \ldots, \emptyset, 001, \emptyset, \ldots, \emptyset) & R_2 &= (\emptyset, \ldots, \emptyset, 1, \emptyset, \ldots, \emptyset) \\
D_3 &= (\emptyset, \ldots, \emptyset, 01, \emptyset, \ldots, \emptyset) & R_3 &= (\emptyset, \ldots, \emptyset, 011, \emptyset, \ldots, \emptyset) \\
D_4 &= (\emptyset, \ldots, \emptyset, 1, \emptyset, \ldots, \emptyset) & R_4 &= (\emptyset, \ldots, \emptyset, 010, \emptyset, \ldots, \emptyset).
\end{aligned}$$

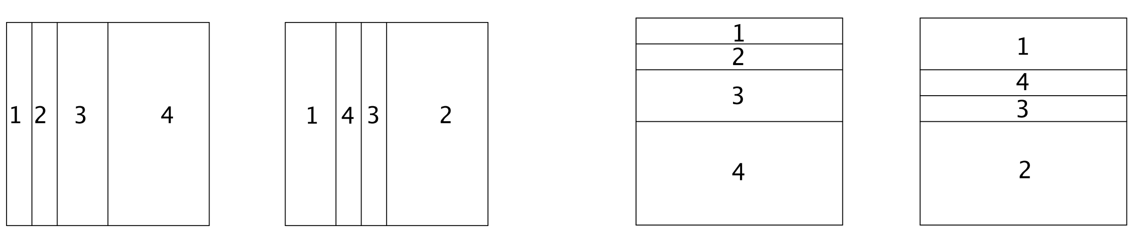
The elements a_1 and a_2 in the $2V$ case are shown in Figure 4.3.

Consider a point x whose k-th coordinate begins with a 1. Then xa_k has all coordinates the same as x except for the k-th one, which is now starts with 01. As a_k only changes values of the k-th coordinate, we will only refer to this coordinate for the rest of this discusion. Each further application of a_k changes the prefix 01 to the prefix 011. In particular, xa_k^m begins with a 0 for all $m > 0$. Considering negative powers of a_k, observe xa_k^{-1} changes the prefix from 1 to 001. Each further application changes the prefix 00 to 000. Thus, xa_k^m begins with a 0 for all $m \neq 0$, and hence a_k has infinite order.

Note that if $k \neq j$, a_k and a_j commute since a_k only changes the k-th coordinate, a_j only changes the j-th coordinate, and that each only considers the entry in its relevant coordinate to decide how to act. Hence, the order of application of the functions does not change the outcome. Further, if $\alpha = a_1^{r_1} \dots a_n^{r_n} = 1$, then in particular α does not act on the i'th coordinate hence $r_i = 0$. This is true for each of the n coordinates. Thus, $\langle A \rangle \cong \mathbb{Z}^n$.

We now define $b \in nV$. Again, the bijection will be given in the subscripts. In both the domain and the range there will be $4(2^n - 1) = 2^{n+2} - 4$ n-rectangles in the partition. There are $2^n - 1$ n-rectangles of the form $(a_1, \dots, a_n)$ where $a_i \in \{0, 1\}$ and $\sum a_i < n$. Each one will be associated with three sub-n-rectangles sitting in the n-rectangle in which every coordinate starts with a one, and these three will all have the same first coordinate. These will be numbered in a similar way to the way the a_k's were above, so all nonzero powers of b will take the larger n-rectangles to a thin slice of $(1, \dots, 1)$

More precisely, consider $u_1, \dots, u_n$ where $u_i \in \{0, 1\}$, with at least one u_i not 1. Consider $m = u_1 u_2 \cdots u_n$, an integer between 0 and $2^n - 2$, written in binary. Then



(a) The element a_1

(b) The element a_2

Figure 4.3: The generators for the $\mathbb{Z}^2$ subgroup in the proof of Theorem 25 when $n = 2$.

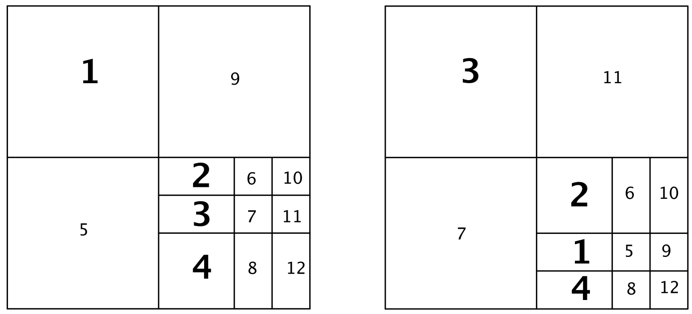


Figure 4.4: The element b from the proof of Theorem 25 in the case $n = 2$.

we define:

$$
\begin{aligned}
D_{4m+1} &= (u_1, \dots, u_n) & R_{4m+1} &= (1^{m+1}0, 110, 1 \dots, 1)\\
D_{4m+2} &= (1^{m+1}0, 100, 1 \dots, 1) & R_{4m+2} &= (1^{m+1}0, 10, 1 \dots, 1)\\
D_{4m+3} &= (1^{m+1}0, 101, 1 \dots, 1) & R_{4m+3} &= (u_1, \dots, u_n)\\
D_{4m+4} &= (1^{m+1}0, 11, 1 \dots, 1) & R_{4m+4} &= (1^{m+1}0, 111, 1 \dots, 1).
\end{aligned}
$$

We left the case when $m = 2^n - 1$ separate only because of notational considerations; the idea is the same. In this case, we have

$$
\begin{aligned}
D_{4m+1} &= (u_1, \dots, u_n) & R_{4m+1} &= (1^{m+2}, 110, 1 \dots, 1)\\
D_{4m+2} &= (1^{m+2}, 100, 1 \dots, 1) & R_{4m+2} &= (1^{m+2}, 10, 1 \dots, 1)\\
D_{4m+3} &= (1^{m+2}, 101, 1 \dots, 1) & R_{4m+3} &= (u_1, \dots, u_n)\\
D_{4m+4} &= (1^{m+2}, 11, 1 \dots, 1) & R_{4m+4} &= (1^{m+2}, 111, 1 \dots, 1).
\end{aligned}
$$

It is routine to verify that the set of all D_i's partition cube, and the same for the set of all R_i's. Figure 4.4 shows the element b in the $2V$ case.

Let $B = \langle b \rangle \cong \mathbb{Z}$. We now show that $\langle A, B \rangle = \langle A \rangle * \langle B \rangle$ which would finish the proof. By definition, nV acts on $\mathfrak{C}^n$. Set X to be the n-rectangle of elements whose first letter is 1 in all coordinates. Set Y to be the union of all n-rectangles with exactly one digit in each coordinate and with at least one coordinate consisting of a 0. (Note this is the complement of X.) Let $1_A \neq w \in \langle A \rangle$. As mentioned above, the generators of A commute, so we can assume $w = a_{k_l}^{p_l} \ldots a_{k_1}^{p_1}$ where $k_i < k_{i+1}$ and $p_i \neq 0$. Let $U' = (u'_1, \ldots, u'_n)$ be the n-rectangle defined by $u'_j = 0$ if $j = k_i$ for some i and 1 otherwise. Thus, $Xw \subset U' \subset Y$.

It remains to show $Yb^n \subset X$ for $n \neq 0$. Note that Y is the union of the $2^n - 1$ n-rectangles that have a 0 as a prefix in some coordinate. By construction, each of these n-rectangles is mapped into X by any nonzero power. Thus, all of Y is as well.

$\square$

4.2.4 Wreath Products, Pre-wreath Structures, and an Embedding Result

The wreath product was discussed in Section 2. There is an analogous idea to the Ping-Pong Lemma to sometimes detect if a group is a (standard restricted) wreath product defined by Brin in [10]. We will explain this technology, and then use it to show that for all $n \in \mathbb{N}$, the group $\mathbb{Z}^n \wr \mathbb{Z}^n$ embeds into nV.

Recall that if Z is a set and G is a subgroup of $\mathrm{Sym}(Z)$, the group of all bijections from Z to itself, then we call (G, Z) a permutation group. A *pre-wreath stucture* is a quadruple (Z, Y, H, X) where H is a non-trivial group, X, Y, and Z are sets, and the following five conditions are satisfied:

(1) $H \leq \mathrm{Sym}(Z)$;

(2) Supp $(H) \subseteq Y \subseteq Z$;

(3) $\emptyset \neq X \subseteq Y$;

(4) For all $h \in H$, we have that $Xh \cap X \neq \emptyset$ implies $h|_X = 1|_X$;

(5) For all $1 \neq h \in H$, there is a $j \in H$ so that $Xjh \neq Xj$.

We call the collection of sets $XH = \{Xh | h \in H\}$ the *carrier* of the structure (Z, Y, H, X). For our purpose, the most important aspect of this idea is the result labeled Proposition 2.5 in [10] repeated here.

Lemma 26. *(Brin, 05) Let (Z, Y, H, X) and (Z, X, G, W) be pre-wreath structures. Then*

1. *$(Z, Y, \langle G, H \rangle, W)$ is a pre-wreath structure;*

2. *The carrier of $(Z, Y, \langle G, H \rangle, W)$ is $WGH = \{Wgh | g \in G, h \in H\}$;*

3. *The permutation group $(\langle G, H \rangle, WGH)$ is isomorphic to the permutation group $(G \wr H, (WG) \times (XH))$.*

The proof of this is not difficult, but is technical and is too long to be included here. We use this lemma as the main tool in the proof of the following theorem.

Theorem 27. *For all $n \in \mathbb{N}$, $\mathbb{Z}^n \wr \mathbb{Z}^n$ embeds into nV.*

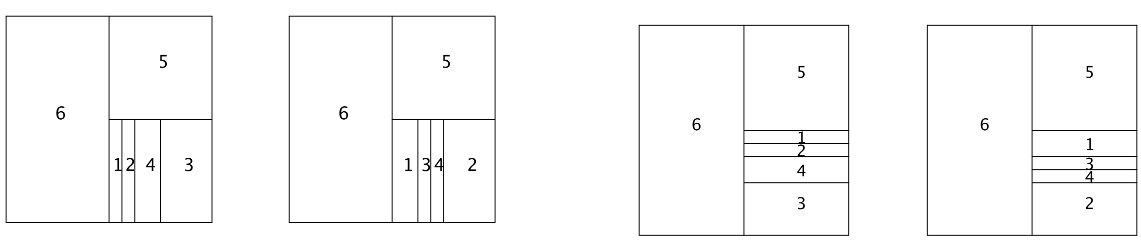
Proof. Fix a positive integer n. We will identify $2n$ elements of nV that will generate a group isomorphic to $\mathbb{Z}^n \wr \mathbb{Z}^n$.

We start with the generators of the bottom group. For an integer $1 \leq k \leq n$, the range and domain patterns for $h_k \in nV$ have four rectangles, with all non-trivial prefixes occurring in the k-th coordinate. Specifically, we have

$$D_1 = (\emptyset, \ldots, \emptyset, 000, \emptyset, \ldots \emptyset) \qquad R_1 = (\emptyset, \ldots, \emptyset, 00, \emptyset, \ldots \emptyset)$$
$$D_2 = (\emptyset, \ldots, \emptyset, 001, \emptyset, \ldots \emptyset) \qquad R_2 = (\emptyset, \ldots, \emptyset, 1, \emptyset, \ldots \emptyset)$$
$$D_3 = (\emptyset, \ldots, \emptyset, 01, \emptyset, \ldots \emptyset) \qquad R_3 = (\emptyset, \ldots, \emptyset, 011, \emptyset, \ldots \emptyset)$$
$$D_4 = (\emptyset, \ldots, \emptyset, 1, \emptyset, \ldots \emptyset) \qquad R_4 = (\emptyset, \ldots, \emptyset, 010, \emptyset, \ldots \emptyset).$$

As no coordinate besides the k-th determines how h_k will act, and it acts by only changing the k-th coordinate, h_k will commute with h_ℓ whenever $k \neq \ell$. Let $X \subset \mathfrak{C}^n$ contain exactly those points with a prefix of 1 in every coordinate. Take any $x \in X$. Then the k-th coordinate of xh_k will have 010 as a prefix. Any further application of h_k to this point will replace the prefix 01 with 011. Hence, for any $m > 0$, the k-th coordinate of xh_k^m will begin with a 0. In particular $xh_k^m \neq x$. Hence, h_k has infinite order. Further, if $\alpha = a_1^{r_1} \ldots a_n^{r_n} = 1$ then $r_1 = \cdots = r_n = 0$ as each generator changes a different coordinate and no coordinate is changed as α is trivial. Thus, $H = \langle h_1, \ldots, h_n \rangle \cong \mathbb{Z}^n$.

Consider $(\mathfrak{C}^n, \mathfrak{C}^n, H, X)$. We will show it is a pre-wreath structure. As $H < nV$, conditions (1) and (2) are satisfied. As X is a nonempty subset of $\mathfrak{C}^n$, condition (3)



(a) The element g_1 (b) The element g_2

Figure 4.5: The generators for the bottom group in the proof of Theorem 27 when $n = 2$.

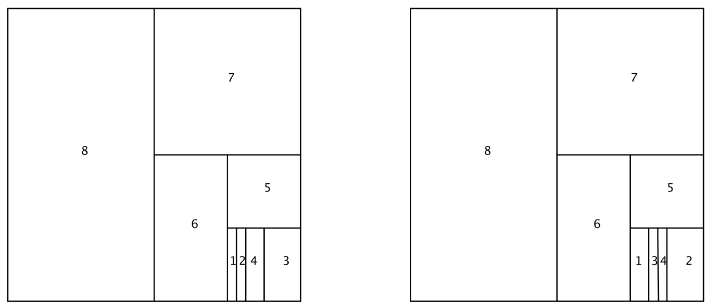


Figure 4.6: The element h_1 from the proof of Theorem 27 in the case $n = 2$. This is one of the generators of the top group. Notice in essence this is the element g_1 shrunk into the lower right quadrant.

is also satisfied. The discussion in the last paragraph shows that X is moved off itself by any positive power of any of the generators of H. A similar exercise will show the same for negative powers. As any $h \in H$ can be written as $h = h_1^{\alpha_1} \cdot h_n^{\alpha_n}$ and each generator acts on a distinct coordinate, we see that any non-trivial $h \in H$ will move X completely off itself. Thus condition (4) is satisfied, and (5) is as well using the identity for j in all cases.

We now look at the generators for the top group. We will simply take the isomorphic copy of everything above, placing a prefix 1 in front of everything. Specifically, for an integer $1 \leq k \leq n$, the range and domain patterns for $g_k \in nV$ have four rectangles containing all the support of the function. All prefixes that are not 1 occur in the k-th coordinate. We have

$$\begin{aligned} D_1 &= (\emptyset, \ldots, \emptyset, 1000, \emptyset, \ldots \emptyset) & R_1 &= (\emptyset, \ldots, \emptyset, 100, \emptyset, \ldots \emptyset) \\ D_2 &= (\emptyset, \ldots, \emptyset, 1001, \emptyset, \ldots \emptyset) & R_2 &= (\emptyset, \ldots, \emptyset, 11, \emptyset, \ldots \emptyset) \\ D_3 &= (\emptyset, \ldots, \emptyset, 101, \emptyset, \ldots \emptyset) & R_3 &= (\emptyset, \ldots, \emptyset, 1011, \emptyset, \ldots \emptyset) \\ D_4 &= (\emptyset, \ldots, \emptyset, 11, \emptyset, \ldots \emptyset) & R_4 &= (\emptyset, \ldots, \emptyset, 1010, \emptyset, \ldots \emptyset). \end{aligned}$$

Let $W \subset \mathfrak{C}^n$ contain exactly those points with a prefix of 11 in every coordinate. The arguments that $G = \langle g_1, \ldots, g_n \rangle \cong \mathbb{Z}^n$ and that $(\mathfrak{C}^n, X, H, W)$ is a pre-wreath structure are virtually identical to the bottom group case.

Thus, by Lemma 26, $\langle G, H\rangle$ is isomorphic to $G \wr H \cong \mathbb{Z}^n \wr \mathbb{Z}^n$.

$\square$

There is a connection between Theorem 27 and the notion of demonstrative groups first discussed in [8]. We will show the connection while deriving a second proof of Theorem 27. This proof will use Proposition 3.6 in [8], which was proved independently and concurrently to the previous material in this section.

If H is a group that acts on a space Y, then we say $G \leq H$ is a *demonstrative group of* H *over* Y if and only if there exists an open set $U \subset Y$ such that for any pair of distinct elements $g_1, g_2 \in G$, we have $Ug_1 \neq Ug_2$. We say that H *acts with local representation* if and only if H acts faithfully on U and for any nonempty open set $U \subset Y$ there is a subgroup $H_U < H$ where H_U is isomorphic to H and Supp $(H_U) \subset U$.

Part 2 of the following proposition refers to the elements $h_1, \ldots, h_n$ defined in the proof of Theorem 27 and the set X also defined there.

Proposition 28. *For all $n \in \mathbb{N}$, the following are true:*

1. *The group nV acts with local representation on $\mathfrak{C}^n$;*

2. *The group $\mathbb{Z}^n = \langle h_1, \ldots, h_n\rangle$ is a demonstrative group of nV over X.*

Proof. For the first statement, let U be any nonempty open set in $\mathfrak{C}^n$. As $\mathfrak{C}^n$ is a finite product of copies of $\mathfrak{C}$, the set of all n-rectangles is a basis. Fix a n-rectangle $B = (B_1, \ldots, B_n) \subset U$. Define nV_U to be the subset of nV containing every element whose support is entirely in B. We need to show nV_U is isomorphic to nV. Let $\Lambda : nV \to nV_U$ as follows. Given $\phi \in nV$, for each n-rectangle $R = (R_1, \ldots, R_n)$ in either the domain pattern or the range pattern for ϕ, $\phi\Lambda$ will have the n-rectangle $(B_1R_1, \ldots, B_nR_1)$ in the same pattern. Extra n-rectangles will be added to complete out the pattern of $\phi\Lambda$, causing it to act trivially outside R. It is straight forward to see that Λ is a well defined group isomorphism.

The second statement was shown in the proof of Theorem 27.

□

We now recall the statement of Proposition 3.6 in [8].

Proposition 29. *Suppose H acts on a space Y with local realization, and that $G \leq H$ is a demonstrative subgroup of H with demonstration set U, then the standard restricted wreath product $H \wr G$ embeds in H.*

Thus we have the following as a consequence.

Corollary 30. *For any n, $nV \wr \mathbb{Z}^n$ embeds into nV.*

Proof. Proposition 28 informs us we can apply Proposition 29 with $H = nV$, $Y = \mathfrak{C}^n$, and $H = \mathbb{Z}^n$, obtaining the result.

□

Theorem 27 now follows immediately. The latter approach does prove a stronger result. However, quoting [8] immediately after the proof of Proposition 3.6:

> We note in passing that it not easy to find a non-trivial wreath product as a demonstrative subgroup; it is difficult to find an open set which will move entirely off itself under the action of all possible non-trivial elements in the base group.

Thus, we feel that the first proof of Theorem 27 has some merit on its own.

Bibliography

[1] Edward A Azoff and Eugen J Ionascu, *Wandering sets for a class of borel isomorphisms of [0, 1)*, Journal of Fourier Analysis and Applications **6** (2000), no. 6, 623–638.

[2] James Belk and Francesco Matucci, *Conjugacy and dynamics in Thompsons groups*, Geometriae Dedicata (2007), 1–23.

[3] Jean-Camille Birget, *The groups of Richard Thompson and complexity*, International Journal of Algebra and Computation **14** (2004), no. 05n06, 569–626.

[4] Collin Bleak, *A geometric classification of some solvable groups of homeomorphisms*, Journal of the London Mathematical Society **78** (2008), no. 2, 352–372.

[5] Collin Bleak, Hannah Bowman, Alison Gordon, Garrett Graham, Jacob Hughes, Francesco Matucci, and Jenya Sapir, *Centralizers in R. Thompson's group V_n*, arXiv preprint arXiv:1107.0672 (2011).

[6] Collin Bleak, Martin Kassabov, and Francesco Matucci, *Structure theorems for groups of homeomorphisms of the circle*, International Journal of Algebra and Computation **21** (2011), no. 06, 1007–1036.

[7] Collin Bleak and Daniel Lanoue, *A family of non-isomorphism results*, Geometriae Dedicata **146** (2010), no. 1, 21–26.

[8] Collin Bleak and Olga Salazar-Díaz, *Free products in R. Thompsons group V*, Transactions of the American Mathematical Society (2013).

[9] Matthew G Brin, *Higher dimensional Thompson groups*, Geometriae Dedicata **108** (2004), no. 1, 163–192.

[10] ______, *Elementary amenable subgroups of R. Thompson's group F*, International Journal of Algebra and Computation **15** (2005), no. 04, 619–642.

[11] Matthew G Brin and Craig C Squier, *Groups of piecewise linear homeomorphisms of the real line*, Inventiones mathematicae **79** (1985), no. 3, 485–498.

[12] Kenneth S Brown, *Finiteness properties of groups*, J. Pure Appl. Algebra **44** (1987), no. 1-3, 45–75.

[13] James W Cannon, William J Floyd, and Walter R Parry, *Introductory notes on Richard Thompson's groups*, Enseignement Mathématique **42** (1996), 215–256.

[14] Pierre de La Harpe, *Topics in geometric group theory*, University of Chicago Press, 2000.

[15] Victor Sergeevich Guba and Mark Valentinovich Sapir, *On subgroups of R. Thompson's group F and other diagram groups*, Sbornik: Mathematics **190** (1999), no. 8, 1077.

[16] Graham Higman, *Finitely presented infinite simple groups*, Department of Pure Mathematics, Department of Mathematics, IAS, Australian National University, 1974.

[17] Derek F Holt, Sarah Rees, Claas E Röver, and Richard M Thomas, *Groups with context-free co-word problem*, Journal of the London Mathematical Society **71** (2005), no. 3, 643–657.

[18] Derek F Holt and Claas E Röver, *Groups with indexed co-word problem*, International Journal of Algebra and Computation **16** (2006), no. 05, 985–1014.

[19] J Lehnert and P Schweitzer, *The co-word problem for the Higman-Thompson group is context-free*, Bulletin of the London Mathematical Society **39** (2007), no. 2, 235–241.

[20] David E. Muller and Paul E. Schupp, *Groups, the theory of ends, and context-free languages.*, J. COMP. AND SYS. SCI. **26** (1983), no. 3, 295–310.

[21] David E Muller and Paul E Schupp, *The theory of ends, pushdown automata, and second-order logic*, Theoretical Computer Science **37** (1985), 51–75.

[22] Olga Patricia Salazar-Díaz, *Thompson's group V from a dynamical viewpoint*, Ph.D. thesis, Binghamton University, 2006.

[23] ———, *Thompson's group V from a dynamical viewpoint*, International Journal of Algebra and Computation **20** (2010), no. 01, 39–70.

[24] Richard J Thompson, *Embeddings into finitely generated simple groups which preserve the word problem*, Studies in Logic and the Foundations of Mathematics **95** (1980), 401–441.

[25] RJ Thompson and R McKenzie, *An elementary construction of unsolvable word problems in group theory*, Word problems, Conference at University of California, Irvine, 1969.