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## ***Physics, Chapter 2: Motion of a Particle (Kinematics)***

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## 2

# Motion of a Particle (Kinematics)

### 2-1 Motion Is Relative

Normally, when we say an object is at rest, we mean that it is at rest with respect to the surface of the earth; when we say a car is moving at a speed of 40 mi/hr, we imply that the motion is taking place at this speed relative to the road. A boat sailing on the river moves with respect to the river's banks, but it also moves with respect to the flowing water in the river. The lift on the wings of an airplane is generated by the motion of the airplane through the air, but it is quite important to know the plane's motion with respect to the ground. When we speak of the motion of a car or a train, we normally mean the motion with respect to the ground, but when we speak of the rated speed of an airplane, we refer to its motion with respect to the air. To avoid confusion in the discussion of motion, it is important to refer the motion to a *frame of reference*, usually thought of as fixed on the earth or fixed relative to the stars, in which the motion is measured. For many problems it is convenient to use moving frames of reference; it is then necessary to specify the nature of the motion of the frame. The frame of reference generally takes the form of a set of coordinate axes in which the motion is pictured.

### 2-2 Uniform Motion in a Straight Line

The simplest type of motion is that in which a body traverses equal distances along a straight line in equal time intervals; this type of motion is called *uniform motion in a straight line*. The *speed* of the body is defined as the *distance traversed divided by the time elapsed*; that is,

$$\text{Speed} = \frac{\text{distance}}{\text{time}},$$

or, in symbols,

$$v = \frac{s}{t}, \quad (2-1)$$

where  $v$  is the speed of the body, and  $s$  is the distance traversed in time  $t$ .

Whenever a number is used to specify the speed of a body, it must always be accompanied by the appropriate units such as feet per second, usually written as ft/sec, or miles per hour (mi/hr), or meters per second (m/sec), or any other appropriate units of distance and time.

One other aspect of motion is the direction in which it takes place. When we wish to specify that a body has moved from a point  $A$  to a point  $B$ , we can use a vector directed from  $A$  to  $B$ ; this vector is the *displacement* of the body. If  $s$  is the distance from  $A$  to  $B$ , the displacement is  $\mathbf{s}$  (printed in boldface type), a vector drawn from  $A$  to  $B$ .

To specify both the speed of a body and its direction of motion, we use the term *velocity*. The *velocity*  $\mathbf{v}$  of a body in uniform motion in a straight line is defined as the displacement divided by the time during which the displacement occurred, or, in symbols,

$$\mathbf{v} = \frac{\mathbf{s}}{t} \quad (2-2)$$

The direction of the velocity is the same as that of the displacement. Velocity is thus a vector quantity. For example, if a train is moving due west with a uniform speed of 60 mi/hr, its velocity is 60 mi/hr west. No statement about the velocity of a body is complete without specifying *both* magnitude and direction.

### 2-3 Relative Velocities

It is frequently important to be able to determine the velocity of a body with respect to one frame of reference when its velocity has been determined with respect to a second frame of reference which is in motion with respect to the first one. For example, the velocity of a ship relative to the water can easily be measured, but what is usually desired is its velocity relative to the shore, for the water is generally in motion.

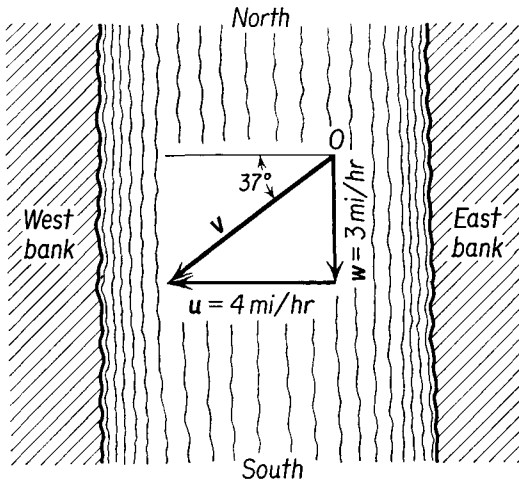
To understand how these velocities are related to each other, let us consider the case of a boat in the river in which the water is moving downstream with a velocity  $\mathbf{w}$  relative to its banks. Let us assume that the boat, if left free, would float downstream with the current; that is, its velocity relative to the water would be zero, but its velocity relative to the banks would be the same as that of the water. Suppose now that the engines of the boat are started and that the boat moves with a velocity  $\mathbf{u}$  relative to the water. Its velocity  $\mathbf{v}$  relative to the banks will therefore be the resultant of the two velocities—the velocity  $\mathbf{w}$  which it acquires because it is moving with the water, and the velocity  $\mathbf{u}$  which it acquires relative to the water.

In the form of an equation,

$$\mathbf{v} = \mathbf{w} + \mathbf{u}.$$

(2-3)

The three quantities involved are vector quantities, and the addition must be performed by vector methods. As a simple illustration, suppose that the velocity of the current in a river is 3 mi/hr south and that a boat heads toward the west with a velocity of 4 mi/hr with respect to the water,



**Fig. 2-1** A boat headed west in a river which flows south actually travels south of west.

as shown in Figure 2-1. It is desired to find the velocity of the boat relative to the shore. The velocity  $\mathbf{u}$  of the boat relative to the water is added vectorially to the velocity  $\mathbf{w}$  of the water relative to the shore to get the resultant velocity  $\mathbf{v}$  of the boat relative to the shore. Its value is 5 mi/hr directed at an angle of  $37^\circ$  south of west.

*Illustrative Example.* Consider the case of an airplane which has to fly from New York to Montreal, due north. Suppose that its normal flying speed in still air is 200 mi/hr. During the flight there is a steady northwest wind of 40 mi/hr. In what direction should the airplane be headed in order to go due north? What will be its speed relative to the ground?

To solve the problem we draw a vector  $\mathbf{w}$  to an appropriate scale, representing the magnitude and direction of the wind, as shown in Figure 2-2. To find the heading of the airplane which will result in a displacement due north, we first draw a straight line due north from  $O$ , the tail of vector  $\mathbf{w}$ . From the head of vector  $\mathbf{w}$ , we swing an arc of radius  $u$  to intersect this line. The vector  $\mathbf{v}$  from  $O$  due north to the arc represents the resultant velocity of the airplane with respect to the ground; its magnitude is the ground speed. The desired heading of the airplane is given by the vector  $\mathbf{u}$ , as shown on the figure. From the figure, if  $u = 200$

mi/hr, and  $w = 40$  mi/hr, then, to the same scale,  $v = 170$  mi/hr. The airplane must be headed about  $8^\circ$  west of north.

**2-4 Instantaneous Speed and Velocity**

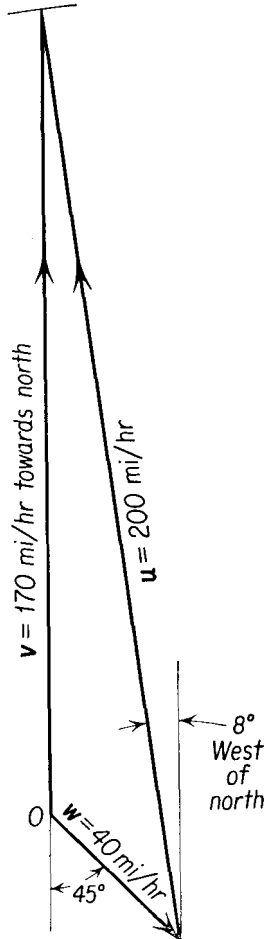
We have thus far confined our discussion to the simplest type of motion, that with constant velocity. Of very great interest is the motion of a body in which its velocity changes. Since velocity is a vector quantity, a change in velocity will occur whenever (a) the speed of the body changes while its direction remains the same, (b) the direction of motion changes while its speed remains the same, or (c) its speed and the direction of its motion change simultaneously. Whenever the *velocity* of a body *changes* in any manner whatever, the motion of the body is said to be *accelerated*.

In order to be able to discuss accelerated motion, it is important to know how to specify the speed and the velocity of the body at any instant or at any point in its path. Suppose that the motion takes place along the  $x$  axis. The *average speed* of the motion is defined as the *distance traversed divided by the elapsed time*. If the object is at  $x_1$  at a time  $t_1$ , and then is at  $x_2$  at a subsequent time  $t_2$ , the average speed  $\bar{v}$  (read  $v$  bar) may be defined in the form of an equation as follows:

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2-4}$$

*Illustrative Example.* Find the average speed with which a car travels down a straight highway 100 mi long if its speed during the first 50 mi is 25 mi/hr and during the second 50 mi, is 75 mi/hr.

Our first reaction may be to say that the average speed is 50 mi/hr, but this is incorrect. To find the average speed, the distance must be divided by the



**Fig. 2-2** Relative velocities of an airplane,  $v$  with respect to the earth and  $u$  with respect to the air, when there is a wind of velocity  $w$  from the northwest.

elapsed time. The distance traversed is 100 mi. The elapsed time is 2 hr for the first 50 mi, and  $\frac{50}{37.5}$  hr for the second 50 mi. The total elapsed time is  $2\frac{2}{3}$  hr.

$$\bar{v} = \frac{100 \text{ mi}}{2.67 \text{ hr}} = 37.5 \frac{\text{mi}}{\text{hr}}.$$

The instantaneous speed at any point  $P$  in the path of a body moving in a straight line, say the  $x$  direction, can be found by taking the average speed during a short time interval  $\Delta t = t_2 - t_1$  during which the particle

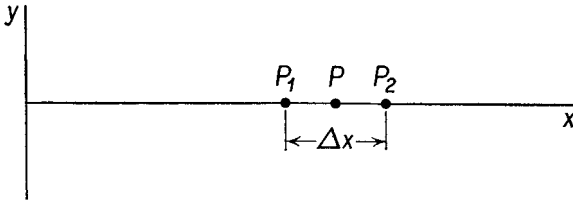


Fig. 2-3

has moved a distance  $\Delta x$  from  $P_1$  to  $P_2$ , points on either side of  $P$  (see Figure 2-3). The average speed of the body is  $\Delta x/\Delta t$ . As the time interval  $\Delta t$  is made shorter, the points  $P_1$  and  $P_2$  close in on point  $P$ , and, in the language of the calculus, we study the value of the average speed over a sequence of *nested* intervals. The sequence of values of the average speed approaches a constant value called a *limit*; this limit is the instantaneous speed  $v_x$  of the body at point  $P$ ; thus

$$v_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{2-5}$$

As in the calculus, the symbol  $\Delta$  (delta) has been used to indicate small increments. Thus if the position may be described as a function of time,

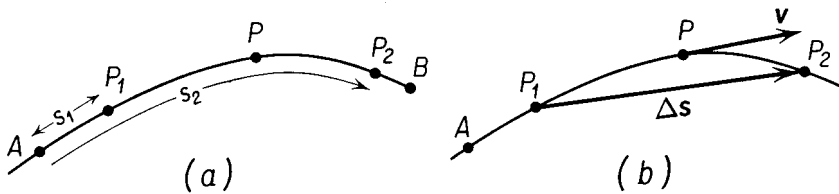


Fig. 2-4 Displacement in curvilinear motion.

the instantaneous speed is the derivative of the position with respect to time. In the case of *uniform* motion in a straight line, the average speed is the same as the instantaneous speed.

Consider the motion along a curved path, sometimes called *curvilinear* motion, such as a motion along  $APB$  of Figure 2-4. The average speed of

the body is again the distance traversed divided by the elapsed time. The distance traveled along the curved path is denoted by  $s$ , measured from some arbitrary reference point  $A$  on the path. If  $s_1$  is the distance of  $P_1$  from  $A$  and  $s_2$  the distance of  $P_2$  from  $A$ , the average *speed* from  $P_1$  to  $P_2$  [Figure 2-4(a)] will be given by

$$\bar{v} = \frac{s_2 - s_1}{t_2 - t_1}, \quad (2-6)$$

where the body passes point  $P_1$  at time  $t_1$  and point  $P_2$  at time  $t_2$ .

If we let

$$\Delta s = s_2 - s_1$$

and

$$\Delta t = t_2 - t_1,$$

then

$$\bar{v} = \frac{\Delta s}{\Delta t}.$$

The instantaneous speed  $v$  may then be determined by a limiting process as

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \quad (2-7)$$

The average *velocity* over the interval  $P_1P_2$  is the vector displacement divided by the time, rather than the scalar distance divided by the time. Calling the displacement of the particle from  $P_1$  to  $P_2$  by  $\Delta \mathbf{s}$ , and the time interval for performing this displacement  $\Delta t$ , the average velocity in the neighborhood of  $P$  becomes

$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{s}}{\Delta t}. \quad (2-8)$$

The displacement  $\Delta \mathbf{s}$  will not, in general, coincide with the actual path from  $P_1$  to  $P_2$ , but as the two points are taken closer and closer to  $P$ , the displacement practically coincides with the actual path along the curve. The direction of the displacement is then tangent to the path at  $P$ . The magnitude of the instantaneous velocity  $\mathbf{v}$  at  $P$  is the instantaneous speed  $v$  at  $P$ , and its direction is tangent to the path at  $P$ , as shown in Figure 2-4(b). The instantaneous velocity may be written as

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{s}}{\Delta t} = \frac{d\mathbf{s}}{dt}. \quad (2-9)$$

Equations (2-8) and (2-9) are *vector* equations. Both sides of the equations contain vector quantities. All equations in physics must relate the same kinds of things. The two sides of the equation must not only

have the same dimensions but must also relate quantities of the same character.

We can visualize the meaning of the vector Equations (2-8) and (2-9) more easily by considering a case in which the displacement  $\Delta \mathbf{s}$  is entirely in the  $x$  direction. Since the velocity vector is parallel to the displacement, the velocity vector must be in the  $x$  direction. Writing  $\mathbf{v}_x$  for velocity in the  $x$  direction, and  $\mathbf{x}$  for displacement in the  $x$  direction, we substitute in Equation (2-9) to obtain

$$\mathbf{v}_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{x}}{\Delta t} = \frac{d\mathbf{x}}{dt}. \quad (2-10)$$

But now there can be no confusion about the direction of the displacement or about the direction of the velocity, since all directions are parallel to the  $x$  axis. Hence we may rewrite Equation (2-10) as a scalar equation in algebraic quantities.

$$v_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \quad (2-11)$$

By considering cases in which the displacement is entirely in the  $y$  direction, and in which the displacement is entirely in the  $z$  direction, we obtain

$$v_y = \frac{dy}{dt}, \quad (2-12)$$

and 
$$v_z = \frac{dz}{dt}. \quad (2-13)$$

Any displacement in space may be thought of as the vector sum of its three component displacements, one in the  $x$  direction, one in the  $y$  direction, and one in the  $z$  direction. Hence Equations (2-11), (2-12), and (2-13) are true in the general case where the motion is in any direction, not just parallel to one of the coordinate axes. The velocity component in the  $x$  direction  $v_x$  depends only on the rate of change of the  $x$  coordinate of the position of the moving object. This conception of the separability of the components of the motion will greatly simplify the study of the motion of a particle, as we shall see when we study the motion of projectiles later in this chapter.

## 2-5 Acceleration

The discussion of motion with varying velocity can best be dealt with in a quantitative manner by the introduction of the concept of acceleration. *The acceleration of a body is defined as the change in its velocity divided by*



the time in which the change takes place. Just as in the case of velocity, we must distinguish between average acceleration and instantaneous acceleration. If the initial velocity of a body is  $\mathbf{v}_1$  at a time  $t_1$ , and the final velocity is  $\mathbf{v}_2$  at a time  $t_2$ , the average acceleration is, from the definition,

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}. \quad (2-14)$$

The instantaneous acceleration is arrived at by examining the average acceleration obtained in a sequence of nested intervals converging on the point  $P$  where the acceleration is to be determined. Referring to Figure

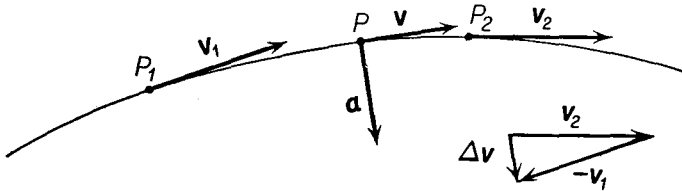


Fig. 2-5 Instantaneous acceleration  $\mathbf{a}$  at a point  $P$ .

2-5, if the instantaneous velocity at point  $P_1$  is given by  $\mathbf{v}_1$ , and the instantaneous velocity at  $P_2$  is  $\mathbf{v}_2$ , the instantaneous acceleration at point  $P$  is given by

$$\mathbf{a} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}. \quad (2-15)$$

If the acceleration is constant, the average acceleration and the instantaneous acceleration are the same.

Let us consider a case in which the motion is entirely in the  $x$  direction. Writing  $a_x$  for the magnitude of the acceleration,  $v_x$  for the speed, and substituting in Equation (2-15), we find

$$a_x = \frac{dv_x}{dt}; \quad (2-16)$$

and for motions entirely in the  $y$  and  $z$  directions we obtain

$$a_y = \frac{dv_y}{dt}, \quad (2-17)$$

and

$$a_z = \frac{dv_z}{dt}. \quad (2-18)$$

Thus Equations (2-16), (2-17), and (2-18) are true for motion which takes

place in any direction and may be considered as the component forms of Equation (2-15). In the component forms the quantities appearing in the equations are all scalar algebraic quantities. The component forms of the equations are easier to use for the solution of most problems. The vector form is the easier way to remember the equations and to develop further theory.

As an illustration of accelerated motion, we can discuss the motion of an airplane under a variety of conditions. As it prepares to take off, the airplane moves along the runway with increasing speed; if this speed is changing at a constant rate, its acceleration is constant and is in the direction of its motion. This is the simplest type of accelerated motion. At the instant of take-off, the direction of the acceleration changes so that it will have a vertical component upward. After the plane has reached the desired altitude and speed and has leveled off in flight, it continues with constant velocity, that is, with zero acceleration. If, during the flight, the plane makes a turn at constant speed, its motion is again accelerated because the direction of motion is changing. When an airplane is preparing to land, it reduces its speed for safe landing. While the speed is being reduced, its motion is also accelerated; this time the direction of the acceleration is opposite to the direction of motion.

In this chapter we shall limit our discussion to motion with constant acceleration, leaving to later chapters the discussion of several cases of varying acceleration.

## 2-6 Straight-Line Motion under Constant Acceleration

The simplest type of motion with constant acceleration is that in which a body moves in a straight line with a speed which is increasing or decreasing at a *constant* rate. We may choose the direction of the  $x$  axis as along the direction of motion and rewrite Equation (2-14) as

$$\bar{a}_x = a_x = \frac{v_x - u_x}{t_f - t_i}, \quad (2-19)$$

in which  $v_x$  is the speed of the body in the  $x$  direction at the final time  $t_f$ ,  $u_x$  is the speed of the body in the  $x$  direction at the initial time  $t_i$ , and  $a_x$  is the constant acceleration of the body in the  $x$  direction during this time interval.

*Illustrative Example.* An airplane approaching a landing field decreases its velocity from 250 mi/hr to 100 mi/hr in 20 sec. Find the acceleration.

Again we choose the  $x$  axis as the direction of motion. We set  $u_x = 250$  mi/hr at time  $t_i$ ,  $v_x = 100$  mi/hr at  $t_f = 20$  sec  $+ t_i$ , and we write

$$a_x = \frac{100 \text{ mi/hr} - 250 \text{ mi/hr}}{20 \text{ sec}} = \frac{-150 \text{ mi/hr}}{20 \text{ sec}} = -7.5 \frac{\text{mi}}{\text{hr sec}}.$$

Thus the speed of the airplane is decreasing at the rate of 7.5 mi/hr each second during the 20-sec interval. With the aid of the appropriate conversion factors, the above result may also be expressed as

$$a_x = -11 \frac{\text{ft/sec}}{\text{sec}} = -11 \frac{\text{ft}}{\text{sec}^2};$$

that is, the speed of the airplane is decreasing at the rate of 11 ft/sec each second during the 20-sec interval.

The question of positive or negative signs for the acceleration of a body, or for other quantities requiring some indication of their direction, is a matter of convenience. For most purposes it is preferable to adopt some consistent scheme of notation, working all problems in the same systematic way. This matter of approaching problems systematically is of great importance in physics and engineering. Problems which at first seem fiendishly difficult often yield to a persistent and systematic approach. The notation generally preferred is one in which the signs employed follow the usual right-handed rectangular coordinate system. The motion is thought of as occurring within a coordinate frame. The direction of the positive  $x$  axis is then the positive direction for the  $x$  components of the displacement, the velocity, and the acceleration. A motion described by a positive value of the velocity and a negative value of the acceleration is one in which the velocity is directed in the direction of increasing  $x$ , while the acceleration is in the opposite direction.

### 2-7 Equations of Motion for Constant Acceleration

When a body is moving in a straight line with constant acceleration  $a$ , we may derive its equations of motion most simply by the methods of the calculus. Dropping the subscript  $x$  from Equation (2-16), we have

$$\frac{dv}{dt} = a, \tag{2-16}$$

thus  $dv = a dt$ ,

which we may integrate as

$$\int dv = \int a dt,$$

yielding  $v = at + u$ , (2-20)

where  $u$  is a constant of integration. We note that at time  $t = 0$  the speed  $v$  is equal to  $u$ . Thus  $u$  is the initial speed of the body.

From Equation (2-10), the definition of instantaneous velocity, we have

$$v = \frac{dx}{dt}, \quad (2-10)$$

and, substituting from Equation (2-10) into Equation (2-20) for  $v$ , we have

$$\frac{dx}{dt} = at + u,$$

or

$$\int dx = a \int t dt + \int u dt,$$

$$x = \frac{1}{2}at^2 + ut. \quad (2-21)$$

In Equation (2-21) we have set the initial position of the body at the origin  $x = 0$  at time  $t = 0$ , thereby setting the constant of integration equal to zero. Equations (2-20) and (2-21) give the speed and position of the body as a function of time when the body is at the origin moving with speed  $u$  at the initial time  $t = 0$ , if the body is moving under constant acceleration  $a$ .

When a body is moving with constant acceleration in a straight line, its average speed is given by the average of its initial and final speeds. This may be shown by algebraic manipulation of Equations (2-20) and (2-21). We first factor the quantity  $t$  from the right-hand side of Equation (2-21) and write

$$x = (u + \frac{1}{2}at)t.$$

From Equation (2-20) we substitute

$$at = v - u$$

into the above equation to find

$$x = \frac{u + v}{2} t. \quad (2-22)$$

From the definition of the average speed as the distance traversed divided by the elapsed time

$$\bar{v} = \frac{x}{t},$$

we see that

$$\bar{v} = \frac{u + v}{2}; \quad (2-23)$$

that is, the average speed is the average of the initial and final speeds, for the case of constant acceleration.

A useful result may be obtained by eliminating the time  $t$  as a variable from Equations (2-20) and (2-22) to obtain an equation relating the initial speed  $u$ , the final speed  $v$ , and the distance traversed  $x$ . We may rewrite

Equation (2-20), by transposing the quantity  $u$ , as

$$v - u = at. \quad (2-24)$$

Both sides of Equation (2-22) may be multiplied by the quantity  $2/t$  to obtain

$$v + u = \frac{2x}{t}. \quad (2-25)$$

Multiplying the left-hand side of Equation (2-24) by the left-hand side of Equation (2-25) and setting the result equal to the product of the right-hand sides of the two equations yields

$$v^2 - u^2 = 2ax, \quad (2-26)$$

which is the result we have sought.

The principal results of the preceding section may be summarized in the following equations for the case of the motion of a particle with a constant acceleration  $a$ , with the particle starting at the origin with initial speed  $u$  at time  $t = 0$ .

$$v = u + at, \quad (2-27a)$$

$$x = ut + \frac{1}{2}at^2, \quad (2-27b)$$

$$v^2 = u^2 + 2ax, \quad (2-27c)$$

$$x = \bar{v}t, \quad (2-27d)$$

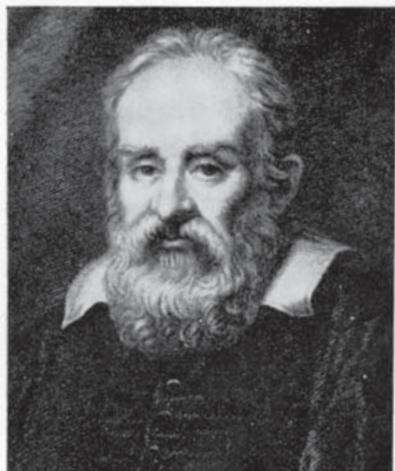
$$\bar{v} = \frac{u + v}{2}. \quad (2-27e)$$

Equations (2-27) are repeated for emphasis, for they will be applied to problems and derivations many times throughout this text.

## 2-8 Freely Falling Bodies

One of the most common examples of motion with constant acceleration is that of a body which is dropped from any height and allowed to fall freely under the influence of gravity. By *free fall* we mean that such effects as air resistance (drag) or lift are assumed to be negligible. A falling leaf, a dandelion seed, and a glider dropped from some altitude are not freely falling bodies. A good approximation to a freely falling body may be obtained by dropping a round, heavy object. This is the motion that was first studied by Galileo. The results of many different experiments performed under many different conditions show that the acceleration of a freely falling body at any point near the earth's surface is a constant for that particular place and does not depend on the weight of the falling object.

The magnitude of the acceleration of gravity varies slightly with altitude, with latitude, and from point to point on the earth's surface having the same latitude and altitude. This is the subject of considerable geophysical exploration. If, at a certain place, the acceleration of gravity is slightly



**Fig. 2-6** Galileo Galilei (1564–1642). Discovered the laws of motion of freely falling bodies and of bodies moving along inclined planes. Constructed a telescope with which he observed the surface features of the moon and discovered four of the moons of Jupiter; his observations helped establish the validity of the heliocentric theory of the universe. (Courtesy of *Scripta Mathematica*.)

high or slightly low, the geophysicist uses this information to help find buried ore or oil bodies beneath the surface of the ground.

For the purposes of most calculations, the value of the acceleration of a freely falling body, designated by  $g$ , may be taken as

$$g = 32 \frac{\text{ft}}{\text{sec}^2} = 980 \frac{\text{cm}}{\text{sec}^2} = 9.8 \frac{\text{m}}{\text{sec}^2}.$$

The direction of the acceleration of gravity is vertically downward. We shall always use the symbol  $g$  to indicate a positive number, as above, and the direction appropriate for use in the sign conventions of a particular problem will always be explicitly shown as  $+g$  or  $-g$ . For very accurate calculations the value of  $g$  appropriate to a particular place should be used. These values may be found in tables of physical constants. For reference, a few values of  $g$  at different latitudes and at sea level are given in Table 2-1.

A clearer understanding of the significance of acceleration can be obtained from the detailed consideration of the free fall of a baseball that is dropped from the top of a very tall building or from a cliff. When we say that a ball is dropped, we mean that its initial velocity is zero. Since it is accelerated downward at the rate of  $32 \text{ ft/sec}^2$ , at the end of the first second it will have acquired a velocity of  $32 \text{ ft/sec}$  downward. Its average velocity during the first second is  $16 \text{ ft/sec}$ , and the distance traveled during the time is  $16 \text{ ft}$ . At the end of the second second, its downward velocity will

TABLE 2-1 VALUES OF  $g$

Latitude	$g$ in cm/sec <sup>2</sup>	$g$ in ft/sec <sup>2</sup>
0°	977.989	32.0862
30°	979.295	32.1290
45°	980.600	32.1719
60°	981.905	32.2147
90°	983.210	32.2575

have been increased by another 32 ft/sec to 64 ft/sec. At the end of the third second, it will again have acquired an additional velocity increment of 32 ft/sec so that its average velocity during the 3 sec is 48 ft/sec, and the distance traversed is 144 ft. Figure 2-7 shows the positions of the ball at 1-sec intervals and the corresponding velocities.

In our discussion of freely falling bodies, the effect of the air on the motion of a body through it was neglected. This discussion thus presents only a first approximation to the actual motion. In many cases this description is sufficiently accurate. However, when the velocity of the body is very great, such as the velocity of a bullet, or if the body is very small, such as a raindrop, or if the body presents a very large surface, such as a parachute, the resistance of the air plays an important part in determining the motion of the body.

*Illustrative Example.* Suppose that a ball is thrown vertically upward with an initial velocity of 80 ft/sec. Determine (a) how high it will go, (b) what velocity it will have as it moves down past its original point of projection, (c) its position 6 sec after it was thrown upward, and (d) the velocity with which it will be moving at this time.

Let us choose a set of coordinates with the origin  $O$  at the point of projection, and let us take the  $y$  axis as the line of motion (see Figure 2-8). The displacement from the origin will be measured by the  $y$  coordinate of the ball; it will be considered positive above the origin and negative below the origin. The acceleration is downward, and its magnitude is

Time of fall in seconds	Velocity of ball in feet per second		Distance of fall in feet
0	0	●	0
1	32	●	16
2	64	●	64
3	96	●	144
4	128	●	256

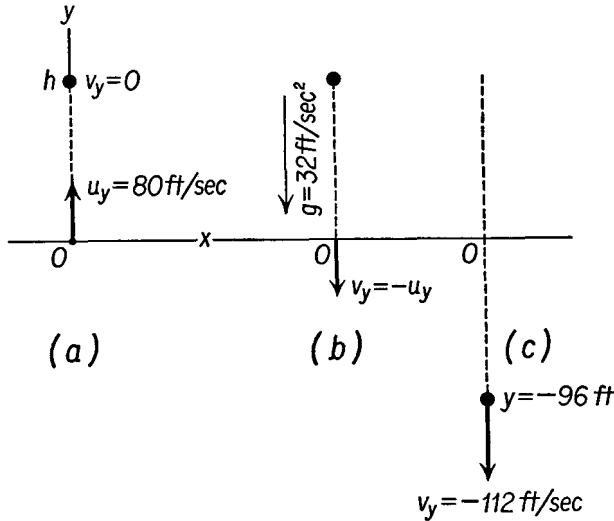
Fig. 2-7 The free fall of a baseball, showing the positions of the ball at intervals of 1 sec and the corresponding velocities.

$g$  at all times and at all points of the path. Rewriting Equations (2-27b) and (2-27a) for motion in the  $y$ -direction, we get

$$y = u_y t + \frac{1}{2} a_y t^2,$$

$$v_y = u_y + a_y t.$$

(a) At the highest point of the path the ball will stop momentarily; this means that  $v_y = 0$ . Other known quantities are  $a_y = -g$  and  $u_y = 80$  ft/sec.



**Fig. 2-8** (a) A ball thrown upward reaches a height  $h$  at which  $v = 0$ . (b) On its return journey it passes the origin with a speed equal to its initial speed but in the opposite direction. (c) Position and velocity of the ball 6 sec after it started its motion.

The unknown quantities are  $y$  and  $t$ . Having two equations in the two unknowns, the problem may be easily solved by the methods of algebra. Making the necessary substitutions,

$$y = 80 \frac{\text{ft}}{\text{sec}} \times t - \frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times t^2,$$

$$0 = 80 \frac{\text{ft}}{\text{sec}} - 32 \frac{\text{ft}}{\text{sec}} \times t.$$

Solving these equations, we find  $t = 2.5$  sec and  $y = 100$  ft. Thus the highest point of the path is 100 ft above the initial point of projection.

(b) When the ball falls down and passes its original point, its position is given by  $y = 0$ . We apply the equations above once again. This time the known quantities are  $y = 0$ ,  $u_y = 80$  ft/sec,  $a_y = -32$  ft/sec<sup>2</sup>. The unknown quantities are  $v_y$  and  $t$ , and once again we have two equations in two unknowns. Making the



necessary substitutions,

$$0 = 80 \frac{\text{ft}}{\text{sec}} \times t - \frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times t^2,$$

$$v_y = 80 \frac{\text{ft}}{\text{sec}} - 32 \frac{\text{ft}}{\text{sec}^2} \times t.$$

From the first equation the solutions are  $t = 0$  and  $t = 5$  sec; that is, the ball is at  $y = 0$  at both of these times. The first of the solutions is trivial; the second one shows that the ball will be back at its starting position 5 sec after it is projected. Substituting the value into the second equation, we find  $v_y = -80$  ft/sec. Thus the ball returns to its initial position with exactly the same speed it had when it started, but in the opposite direction.

(c) and (d) To find the position and velocity of the ball after 6 sec of motion, we again substitute appropriate values into our two equations. This time the known quantities are  $t = 6$  sec,  $u_y = 80$  ft/sec,  $a_y = -32$  ft/sec<sup>2</sup>, and the unknown quantities are  $y$  and  $v_y$ . Once again we have two equations in two unknowns, and a complete algebraic solution is possible. Substituting, we find that

$$y = 80 \frac{\text{ft}}{\text{sec}} \times 6 \text{ sec} - \frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times (6 \text{ sec})^2,$$

$$v_y = 80 \frac{\text{ft}}{\text{sec}} - 32 \frac{\text{ft}}{\text{sec}^2} \times 6 \text{ sec}.$$

The first equation immediately yields  $y = -96$  ft, and the second equation gives  $v_y = -112$  ft/sec. This is the same speed that a body would acquire if it fell from the highest point in the path, a distance of 196 ft.

## 2-9 Motion of a Projectile

The motion of a projectile after it leaves the muzzle of the gun is a special case of a freely falling body in which the initial velocity of the projectile is at any arbitrary direction to the vertical. We shall again limit our discussion to the ideal case in which the air resistance is neglected. Suppose that a bullet is fired horizontally with an initial velocity  $\mathbf{u}$ , as illustrated in Figure 2-9. Once the bullet leaves the muzzle of the gun, its acceleration is vertically downward and is equal to  $\mathbf{g}$  (directed vertically downward). This means that, in addition to its horizontal motion with velocity  $\mathbf{u}$ , the bullet will acquire an additional velocity vertically downward equal to  $\mathbf{g}t$ ; that is, the downward velocity will increase with the time just as if it were dropped. In fact, if a second bullet were dropped at the same time that the first bullet was fired, both would reach the ground at exactly the same time. The actual velocity  $\mathbf{v}$  at any instant will be the vector sum of these two velocities; that is,

$$\mathbf{v} = \mathbf{u} + \mathbf{g}t, \quad (2-28)$$

as shown in Figure 2-9.

To determine the path of the bullet, let us choose a set of rectangular coordinates with the origin  $O$  situated at the muzzle of the gun and choose the  $x$  axis in the horizontal direction. Remembering that there is no acceleration in the  $x$  direction, we find from Equation (2-27b) that

$$x = ut; \tag{2-29}$$

that is, the motion of the projectile in the  $x$  direction is one with constant velocity. Since the initial velocity has no component in the  $y$  direction

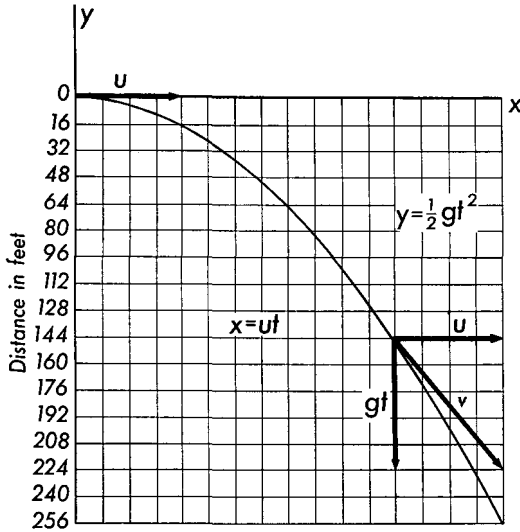


Fig. 2-9 Path of a bullet fired horizontally is a parabola.

$u_y = 0$ . The acceleration is entirely in the  $y$  direction with  $a_y = -g$ . We rewrite Equation (2-27b) with the symbol  $y$  replacing each  $x$  to get the general equation appropriate to vertical motion, and substitute values appropriate to the problem to find

$$y = -\frac{1}{2}gt^2. \tag{2-30}$$

These values of  $x$  and  $y$  are plotted in Figure 2-9, and the curve obtained shows the path of the bullet. To obtain the equation of the path, we eliminate  $t$  from the two equations displayed above to obtain

$$y = \frac{-g}{2u^2}x^2. \tag{2-31}$$

This is the equation for the path, which is a parabola.

*Illustrative Example.* A bomber flying eastward with a velocity of 480 mi/hr drops a bomb from an elevation of 1,600 ft. Assuming that we can neglect

air resistance, determine where the bomb will land and how long it will take to get there.

Since the bomb was in the airplane until the instant it was released, its initial velocity is the same as that of the airplane; that is,  $u_x = 480$  mi/hr = 704 ft/sec. We can obtain the time of fall for the bomb from Equation (2-30). Substituting  $y = -1,600$  ft, we get

$$-1,600 \text{ ft} = -\frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times t^2,$$

from which  $t = 10$  sec.

Using Equation (2-29), we can find where it will strike the ground 10 sec after it was released:

$$x = 704 \frac{\text{ft}}{\text{sec}} \times 10 \text{ sec} = 7,040 \text{ ft.}$$

The bomb will strike at a distance of 7,040 ft east of the point at which it was released. If the plane continues to move with a velocity of 480 mi/hr after dropping the bomb, the latter will strike the target when the plane is directly overhead.

The velocity with which the bomb will strike the ground will be the vector sum of the horizontal velocity

$$u = 704 \frac{\text{ft}}{\text{sec}}$$

and the vertical velocity

$$gt = 320 \text{ ft/sec.}$$

Hence  $v = \sqrt{(704)^2 + (320)^2} \frac{\text{ft}}{\text{sec}} = 772 \frac{\text{ft}}{\text{sec}}$

directed at an angle of about  $22^\circ$  with the horizontal.

While a projectile may be fired in any direction, the only acceleration it experiences is vertical. The projectile follows a parabolic path; hence the motion lies entirely within a plane determined by the vertical and the direction of the barrel of the gun. We can describe the motion most conveniently in two dimensions, considering the  $y$  direction as vertical and the  $x$  direction as the direction of the horizontal projection of the gun barrel. A convenient origin of coordinates is the mouth of the gun barrel. The path of the projectile is shown in Figure 2-10. If the projectile is fired with an initial velocity  $u$ , at an angle of elevation  $\theta$  with the horizontal, the  $x$  component of the initial velocity vector will be

$$u_x = u \cos \theta,$$

and the  $y$  component of the initial velocity vector will be

$$u_y = u \sin \theta.$$

The motion may be completely described from Equations (2-27a) and

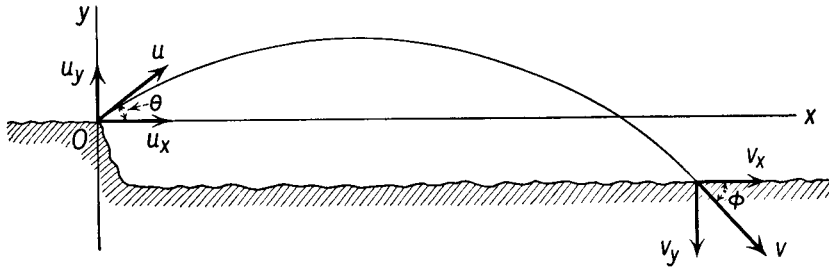


Fig. 2-10 Projectile fired from a cliff.

(2-27b), giving the  $x$  and  $y$  positions and velocity as functions of time. Re-writing these equations under the conditions of the problem, with  $u_x = u \cos \theta$ ,  $u_y = u \sin \theta$ ,  $a_x = 0$ ,  $a_y = -g$ , we have

$$x \text{ position:} \quad x = ut \cos \theta; \quad (2-32a)$$

$$x \text{ velocity:} \quad v_x = u \cos \theta; \quad (2-32b)$$

$$y \text{ position:} \quad y = ut \sin \theta - \frac{1}{2}gt^2; \quad (2-32c)$$

$$y \text{ velocity:} \quad v_y = u \sin \theta - gt. \quad (2-32d)$$

To find the magnitude and direction of the velocity vector  $\mathbf{v}$  at the time  $t$ , we utilize the techniques of Chapter 1:

$$\text{magnitude of } \mathbf{v}: \quad v = (v_x^2 + v_y^2)^{1/2}; \quad (2-32e)$$

$$\text{direction of } \mathbf{v}: \quad \phi = \arctan \frac{v_y}{v_x}; \quad (2-32f)$$

where  $\phi$  is the angle between the velocity vector and the positive  $x$  axis.

*Illustrative Example.* A gun located on a cliff 160 ft high fires a shell with a muzzle velocity of 1,600 ft/sec at an angle of elevation of  $37^\circ$ , as in Figure 2-10. Find the time of flight, the horizontal distance the shell will travel, and the velocity with which the shell will strike the ground.

From the statement of the problem, the shell will strike the ground at a point where  $y = -160$  ft. Other known quantities are  $u = 1,600$  ft/sec,  $\theta = 37^\circ$ . The unknown quantities are  $v_x$ ,  $v_y$ ,  $x$ , and  $t$ . Since there are four unknowns, we require four independent equations for the solution of the problem. Equations (2-32a to d) fulfill the requirements of the problem. With four equations and four unknowns the problem has been reduced to algebra. Putting numerical values in Equation (2-32a) yields

$$x = 1,600 \frac{\text{ft}}{\text{sec}} \times 0.8 \times t;$$

Equation (2-32b) yields

$$v_x = 1,600 \frac{\text{ft}}{\text{sec}} \times 0.8 = 1,280 \frac{\text{ft}}{\text{sec}};$$

Equation (2-32c) yields

$$-160 \text{ ft} = 1,600 \frac{\text{ft}}{\text{sec}} \times 0.6t - \frac{1}{2} \times 32 \frac{\text{ft}}{\text{sec}^2} \times t^2;$$

and Equation (2-32d) yields

$$v_y = 1,600 \frac{\text{ft}}{\text{sec}} \times 0.6 - 32 \frac{\text{ft}}{\text{sec}^2} \times t.$$

Solving Equation (2-32c) for  $t$ , we get

$$t^2 - 60t - 10 = 0,$$

so that

$$t = \frac{60 \pm \sqrt{3,600 + 40}}{2},$$

from which

$$t = 60.2 \text{ sec.}$$

The negative value of  $t$  obtained in this solution has been discarded as physically meaningless. With this result, the other parts of the problem yield

$$x = 1,600 \frac{\text{ft}}{\text{sec}} \times 0.8 \times 60.2 \text{ sec} = 77,000 \text{ ft};$$

$$v_y = 1,600 \frac{\text{ft}}{\text{sec}} \times 0.6 - 32 \frac{\text{ft}}{\text{sec}^2} \times 60.2 \text{ sec} = -966 \frac{\text{ft}}{\text{sec}}.$$

From Equation (2-32e) we find

$$\begin{aligned} v^2 &= v_x^2 + v_y^2 = \left(1,280 \frac{\text{ft}}{\text{sec}}\right)^2 + \left(-966 \frac{\text{ft}}{\text{sec}}\right)^2 \\ &= (1.28 \times 10^3)^2 + (0.966 \times 10^3)^2 \\ &= 2.57 \times 10^6; \end{aligned}$$

$$v = 1.60 \times 10^3 \frac{\text{ft}}{\text{sec}} = 1,600 \frac{\text{ft}}{\text{sec}}.$$

From Equation (2-32f) we find

$$\begin{aligned} \phi &= \text{arc tan } \frac{v_y}{v_x} = \text{arc tan } \frac{-966}{1,280} \\ &= \text{arc tan } (-0.755) \\ &= -37.1^\circ. \end{aligned}$$

The magnitude of  $\mathbf{v}$  and its direction could have been predicted either from the properties of the parabola or from the fact that the shell is a freely falling body once it leaves the gun.

### Problems

2-1. A car is driven over a measured mile in 1.5 min. Determine the speed of the car (a) in miles per hour and (b) in feet per second.

2-2. A river steamer can travel at the rate of 15 mi/hr in still water. How

long will the trip between two cities 60 mi apart take (a) downstream and (b) upstream, if the river current is 3 mi/hr? (c) What will be the average speed for the round trip?

2-3. A ferryboat which can sail at the rate of 8 mi/hr in still water travels straight across a river  $\frac{1}{2}$  mi wide in which there is a current of 2 mi/hr. (a) What is the velocity of the ferryboat with respect to the shore? (b) How long does a trip take?

2-4. The first three runners in a 100-yd race were clocked in 9.5 sec, 10.0 sec, and 10.5 sec, respectively. (a) What was the average speed of each runner and (b) how far apart were the first and last runners when the first one reached the finish line?

2-5. An airplane heads due north with a velocity of 250 mi/hr. A west wind is blowing with a velocity of 40 mi/hr. What is the velocity of the airplane relative to the ground?

2-6. An airplane whose normal speed in still air is 260 mi/hr must travel due east. (a) What course must the aviator set for the plane when there is a steady southwest wind of 40 mi/hr? (b) How long will it take to travel 850 mi?

2-7. An automobile starting from rest acquires a speed of 40 mi/hr in 12 sec. What is its average acceleration?

2-8. How long will it take for a car, starting from rest, to acquire a speed of 60 mi/hr if its acceleration is 12 ft/sec<sup>2</sup>?

2-9. The brakes are applied to the wheels of a locomotive when it is traveling at 70 mi/hr. It comes to rest 24 sec after the brakes are applied. What is its average acceleration?

2-10. An automobile which is traveling at a speed of 55 mi/hr must be brought to a stop within 150 ft. What is the minimum acceleration that must be given to the car to accomplish this?

2-11. An airplane taking off on a runway 1,200 ft long must acquire a speed of 80 mi/hr to get safely into the air. (a) What is the minimum safe acceleration for this airplane? (b) How long will it take for the airplane to acquire this speed when so accelerated?

2-12. A car approaching a turn in the road has its speed decreased from 50 mi/hr to 30 mi/hr while traversing a distance of 120 ft. (a) What was its acceleration and (b) how long did it take to traverse this distance?

2-13. A boy drops a stone from a bridge 80 ft above the water. (a) With what speed did the stone strike the water? (b) With what speed would the stone have struck the water if it had been thrown down with a speed of 24 ft/sec?

2-14. A boy throws a ball vertically upward and catches it 1 sec later. (a) How high up did the ball go? (b) With what speed was it thrown upward?

2-15. A boy throws a stone horizontally with a speed of 30 ft/sec from a cliff 256 ft high. (a) How long will it take the stone to strike the ground? (b) Where will the stone land? (c) With what velocity will the stone strike the ground?

2-16. A small block starting from rest takes 5 sec to slide down an inclined plane 80 cm long. (a) What was its acceleration and (b) with what speed did it reach the bottom of the incline?

2-17. Two horizontal wires are placed parallel to each other 100 cm apart, one directly above the other. A falling ball is clocked as it passes each of these

wires. If the time elapsed is 0.20 sec, determine the speed the ball had when it passed each wire.

2-18. Fighter planes fly at 35,000 ft elevation. What must be the muzzle velocity of an antiaircraft shell to reach this height, neglecting air resistance?

2-19. A rifle fires a bullet with a speed of 30,000 cm/sec. If the elevation of the rifle is  $30^\circ$  with the horizontal, determine (a) the range of the bullet on horizontal ground and (b) the velocity of the bullet when it reaches the ground.

2-20. A projectile is fired vertically upward with an initial velocity of 1,800 ft/sec. (a) How high does it rise? (b) What velocity will it have 5 sec after leaving the gun? (c) What is its altitude 5 sec after leaving the gun?

2-21. A car moving with a speed of 30 mi/hr reaches the top of a hill. As it goes down the hill, its speed increases to 45 mi/hr in 1.5 min. (a) What is the acceleration of the car and (b) what distance does it travel in this time?

2-22. A stone thrown horizontally from a hill takes 6 sec to reach the ground. Determine, in meters, the height of the hill.

2-23. A falling stone is seen to pass a window 2 m high in 0.3 sec. (a) Determine the average speed of the stone. (b) Determine the speed with which it reaches the level of the top of the window. (c) Determine the height above this point from which it fell.

2-24. The distance between two stop lights on a cross-town street is 800 ft. If the acceleration of a certain car, both positive and negative, is kept at  $6 \text{ ft/sec}^2$ , and if the speed limit on this street is 30 mi/hr, determine the minimum time to traverse this distance.

2-25. A ball is thrown a distance of 65 ft in 1.2 sec. Assuming that it was caught at the same level as it was thrown, (a) determine how high the ball rose in its path of motion. (b) With what velocity was the ball thrown?

2-26. A gun fires a shell with a velocity of 600 m/sec at an angle of  $45^\circ$  with the horizontal. Neglecting air resistance, (a) determine the range of this gun, (b) determine the maximum height reached by the shell, and (c) determine the time of flight of this shell on level ground.

2-27. Derive the equation for the range of a projectile fired on level ground,  $R = \frac{u^2 \sin 2\theta}{g}$ , where  $R$  is the range,  $\theta$  is the angle of elevation, and  $u$  is the initial velocity. Show that the maximum range is achieved when  $\theta = 45^\circ$ .

2-28. The  $x$  coordinate of an object moving along the  $x$  axis is given by the equation  $x = 3 - 5t + 12t^2$  ft. Find the corresponding equation for the velocity and acceleration of the object at any time  $t$ .

2-29. A body moving in space has its motion described by the equations  $x = 12t + 15$ ,  $y = 6t^2$  where the distances are in meters and the time is given in seconds. Find the magnitude and direction of the velocity and the acceleration when  $t = 3$  sec.

2-30. A ball is thrown toward a building 50 ft distant at a speed of 100 ft/sec. At what angle must it be thrown if it is to pass through a window 42 ft from the ground?

2-31. A railroad car is moving due north at a speed of 60 mi/hr. A ball is thrown from the window due east at an angle of elevation of  $30^\circ$  and a speed of 40 ft/sec. (a) Find the time at which it strikes the ground 10 ft below the window

of the car. (b) How far east of the track does the ball land? (c) How far north of the point of projection does the ball land?

2-32. By differentiation with respect to time, show that the equation

$$x = x_i + ut + \frac{1}{2}at^2$$

describes the position of a particle whose initial position is  $x_i$  and whose initial speed is  $u$  moving along the  $x$  axis with constant acceleration  $a$ .

2-33. When a balloon is at a height of 6400 ft and rising at a speed of 32 ft/sec, a stone is thrown vertically out of the balloon. The stone hits the ground directly below in 20 sec. (a) What was the initial velocity of the stone relative to the balloon? (b) Relative to the ground?

2-34. Motorist  $A$ , starting from rest, accelerates at a rate of 6 ft/sec<sup>2</sup>. At the same time that  $A$  begins, motorist  $B$ , starting from rest at a point 100 ft ahead of  $A$ , accelerates at a rate of 4 ft/sec<sup>2</sup>. (a) How far does motorist  $A$  travel before they meet? (b) At the instant they meet each motorist decelerates at the rate of 5 ft/sec<sup>2</sup> until his car comes to rest. How far apart are they when they have stopped?

2-35. A train is moving with uniform speed along a level road. A man on the observation platform drops a ball. What is the path of the ball as observed (a) by the man on the train and (b) by another person standing at a short distance from the tracks?

2-36. In a laboratory experiment an air rifle is clamped in position and aimed by sighting along the barrel. The target is released just as the bullet leaves the muzzle of the rifle. Show that the bullet will always hit the target.

2-37. If there is no wind, raindrops fall vertically with uniform speed. A man driving a car on a windless rainy day observes that the tracks left by the raindrops on the side windows are all inclined at the same angle. Show how the vertical speed of the raindrops can be determined from the inclination of the tracks and the reading of the speedometer.

2-38. Show that the speeds of a projectile are the same at any two points in its path which are at the same elevation.

2-39. A boy seated in a rapidly moving railroad car tosses a ball up into the air. Will the ball come down in front of him; behind him; into his hand? What will happen when the car is accelerating in the forward direction? Going round a curve?