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ANALYTICAL SOLUTION FOR SPEED TO ACHIEVE A DESIRED OPERATING POINT FOR A GIVEN FAN OR PUMP

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TECHNICAL NOTE:

ANALYTICAL SOLUTION FOR SPEED TO ACHIEVE A DESIRED OPERATING POINT FOR A GIVEN FAN OR PUMP

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ABSTRACT. *The Affinity Laws for fans (and pumps) provide a way of determining new fan or pump speed given fan or pump performance curve data and a desired operating point (combination of flow rate and pressure) that does not fall on the curve. However, the affinity law calculations require using a point on the curve (hereafter referred to as the "basic point") to determine the new speed. Most references regarding the Affinity Laws do not give a clear description of the method for determining the "basic point", and improper selection of this point can affect the results considerably. This article describes the requirements for the "basic point," and presents an analytical solution to determine the "basic point" and the desired operating speed for the fan or pump to operate at the desired operating point conditions.*

Keywords. *Affinity laws, Fan curve, Fan laws, Fans, Fan speed, Pump curve, Pump laws, Pumps, Pump speed, Speed, System curve.*

Fans are used to create air flow through a system, such as for grain drying, aeration, or ventilation. System curves describe the resistance (pressure) that must be met to achieve various air flow rates through the system. A sample system curve is shown as a dashed line in figure 1. When desired rates are selected for air flow through the system, the system curve determines the flow rate-pressure combinations for those desired operating points.

Fan or pump performance curves are frequently plotted with pressure (preferably total pressure, although static pressure is sometimes used instead) as a function of flow rate for a particular speed. Performance data for a particular fan operating at 1825 r/min are given in table 1 and depicted in figure 1. When a particular fan operating at a particular speed is used in a system, the operating point (flow rate-pressure combination) occurs where the fan and system curves intersect.

As a desired operating point typically will not fall directly on a fan performance curve, there are times when fan speed can be adjusted so the adjusted fan performance curve at the new speed intersects the system curve at the desired operating point. Henderson et al. (1997) present the following Affinity Laws (also known as fan laws or pump laws) for estimating the performance of geometrically similar fans operating at speeds other than that at which a particular fan performance curve was determined.

$$\frac{Q_1}{Q_2} = \left(\frac{N_1}{N_2}\right)\left(\frac{D_1}{D_2}\right)^3\left(\frac{K_2}{K_1}\right) \quad (1)$$

$$\frac{p_{t1}}{p_{t2}} = \left(\frac{N_1}{N_2}\right)^2\left(\frac{D_1}{D_2}\right)^2\left(\frac{\rho_1}{\rho_2}\right)\left(\frac{K_2}{K_1}\right) \quad (2)$$

$$\frac{P_1}{P_2} = \left(\frac{N_1}{N_2}\right)^3\left(\frac{D_1}{D_2}\right)^5\left(\frac{\rho_1}{\rho_2}\right)\left(\frac{K_2}{K_1}\right) \quad (3)$$

where
Q = volumetric flow rate,
D = fan diameter,
N = fan speed,
K = compressibility factor (dimensionless),
p_t = total pressure,
ρ = air density,
P = fan input power, and
subscripts 1 and 2 relate to different fan (or pump) diameters or operating conditions.

Walski et al. (2003) and Simpson and Marchi (2013) have shown that the affinity laws work well for describing the relationship between pressure and flow rate (eqs. 1 and 2).

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Table 1. Pressure and flow rate data describing the performance curve for a fan operating at 1825 r/min.

Pressure (kPa)	0.00	0.249	0.498	0.747	0.995	1.24	1.49	1.74	1.99	2.24
Flowrate (m^3/min)	680	651	623	595	572	547	524	481	439	354

Simpson and Marchi (2013) also showed that the affinity law for power (eq. 3) assumes that the efficiency of the fan (or pump) does not change as the speed changes. Walski et al. (2003) and Simpson and Marchi (2013) both showed using the affinity laws for calculating power (related to pump efficiency) gave erroneous results, with increasing overestimation of the efficiency as the amount of reduction in fan (or pump) speed increased. Simpson and Marchi (2013) recommended using equation 4 for estimating the efficiency (η_2) of a particular fan or pump (with $D_1 = D_2$) at a new speed (N_2) based on the efficiency (η_1) at the nominal speed (N_1):

$$\eta_2 = 1 - (1 - \eta_1) \cdot \left(\frac{N_1}{N_2} \right)^{0.1} \quad (4)$$

One approach to determining a fan speed at which the fan curve goes through the desired operating point is a graphical trial-and-error approach. "Guessing" a new fan speed of 1750 r/min (using the same fan in similar conditions, $D_1 = D_2$, and $K_1 = K_2$), equations 1 and 2 above can be used with the original data from table 1 to determine the "adjusted" fan performance curve pressures and flow rates. An example (using the data point with pressure of 0.249 kPa and flow rate of 651 m^3/min) of these calculations is given below, with the results shown in table 2 and figure 1.

$$Q_2 = Q_1 \cdot \left(\frac{N_2}{N_1} \right) = 651 \frac{m^3}{min} \left(\frac{1750 \frac{r}{min}}{1825 \frac{r}{min}} \right) = 624 \frac{m^3}{min}$$

$$p_{t2} = p_{t1} \cdot \left(\frac{N_2}{N_1} \right)^2$$

$$= 0.249\ kPa \left(\frac{1750 \frac{r}{\min}}{1825 \frac{r}{\min}} \right)^2$$

$$= 0.229\ kPa$$

As fan speed decreases, the fan curve moves down and to the left on the graph. "Guessing" a new fan speed of 1900 r/min and repeating the analysis gives the results shown in table 3 and in figure 1. As fan speed increases, the fan curve moves up and to the right on the graph. While relatively straightforward, this approach has the drawbacks that multiple trial-and-error iterations may be required, and the accuracy of the desired operating speed determined with

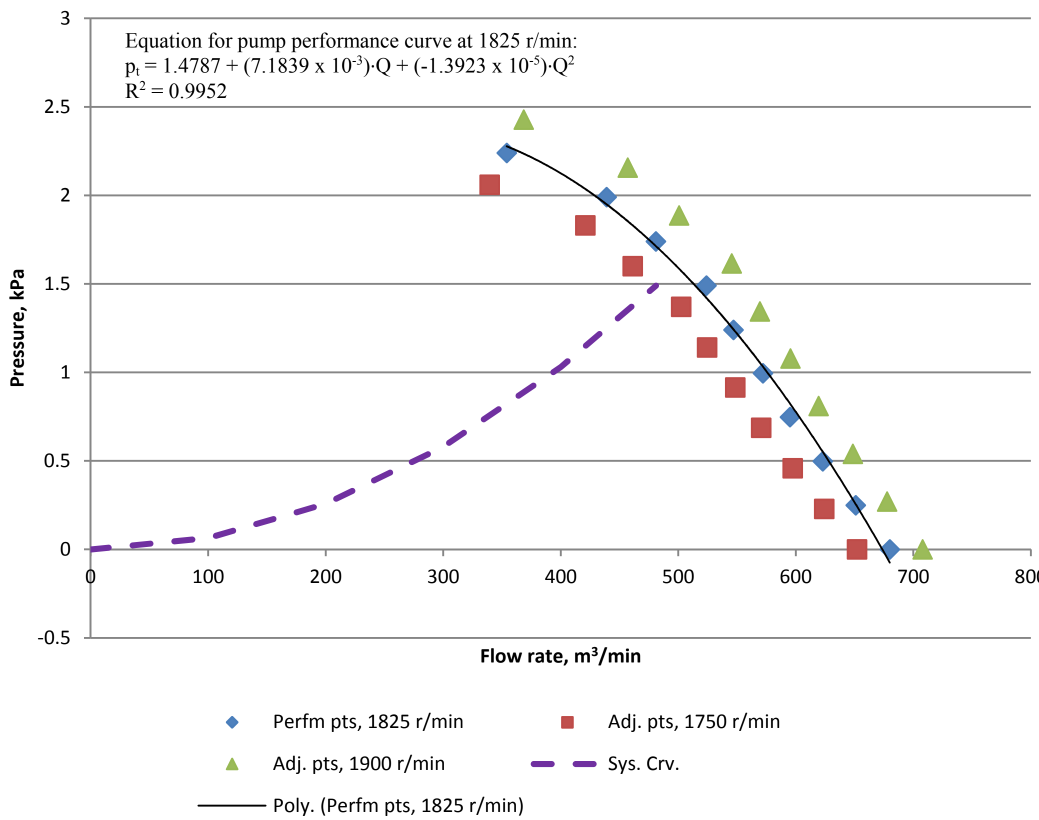

Figure 1. Sample system curve and fan performance curve with performance points measured at a fan speed of 1825 r/min, and the least squares best-fit second-order polynomial for the performance curve. The original fan performance curve data are also shown adjusted to new fan speeds of 1750 r/min (table 2), and 1900 r/min (table 3).

Table 2. Pressure and flow rate data describing the performance curve for the fan from table 1 adjusted to an operating speed of 1750 r/min.

Pressure (kPa)	0.000	0.229	0.458	0.687	0.915	1.14	1.37	1.60	1.83	2.06
Flow rate (m^3/min)	652	624	597	571	548	525	502	461	421	339

Table 3. Pressure and flow rate data describing the performance curve for the fan from table 1 adjusted to an operating speed of 1900 r/min.

Pressure (kPa)	0.000	0.270	0.540	0.810	1.08	1.34	1.61	1.89	2.16	2.43
Flow rate (m^3/min)	708	678	649	619	596	569	546	501	457	369

this approach depends on the resolution of the trial-and-error guesses for the new speed.

The Henderson et al. (1997) reference also describes a procedure and gives an example of an approach to determine the new fan operating speed. Unfortunately, the description and example lack some detail. It appears from equations 1 and 2 that it would be simple to determine the new operating speed for a fan to achieve the new desired operating point. Example 1 will demonstrate that it isn't quite as simple as it first appears. Example 2 will demonstrate that it is possible to obtain an analytical solution for the new fan speed.

OBJECTIVE

The objective of this article is to describe the analysis and present an example of the analytical solution for determining fan (or pump) speed to achieve a desired operating point for a given fan (or pump).

EXAMPLE 1

For this example, we will use the fan performance curve (at 1825 r/min) in figure 1, and a desired operating point (from the system curve shown in fig. 1) with a flow rate Q_d of 481 m^3/min, and a pressure p_{td} of 1.49 kPa. We want to determine the speed at which to operate the fan to achieve the desired operating point. The two data points from the fan performance curve (at the fan speed $N_7 = N_8 = 1825$ r/min) closest to the desired operating point are: $Q_7 = 524$ m^3/min and $p_{t7} = 1.49$ kPa, and $Q_8 = 481$ m^3/min and $p_{t8} = 1.74$ kPa. At these low pressures it is appropriate to use the assumptions that $\rho_1 = \rho_2$, and $K_1 = K_2$. As the same fan is used, $D_1 = D_2$.

Using equation 1 with the simplifying assumptions for ρ and K and point 7, we solve for speed as follows:

$$N_{d1} = N_7 \frac{Q_d}{Q_7} = 1825\,\frac{r}{min}\,\frac{481\,\frac{m^3}{min}}{524\,\frac{m^3}{min}} = 1675.2\,\frac{r}{min}$$

As a double-check, let's calculate the pressure we should get with this speed, remembering that our desired pressure is 1.49 kPa.

$$p_{td} = p_{t7}\left(\frac{N_{d1}}{N_7}\right)^2 = 1.49\,kPa\left(\frac{1675.2\,\frac{r}{\min}}{1825\,\frac{r}{\min}}\right)^2 = 1.26\,kPa$$

We notice a problem as the pressure determined with a fan speed of 1675 r/min (1.26 kPa) is not the desired pressure (1.49 kPa).

Let's go back and use equation 2 with point 7 and the simplifying assumptions to see what results that gives:

$$N_{d2} = N_7\sqrt{\frac{p_{td}}{p_{t7}}} = 1825\,\frac{r}{\min}\sqrt{\frac{1.49\,kPa}{1.49\,kPa}} = 1825\,\frac{r}{\min}$$

We know that isn't going to work, as we already know from the fan curve data (table 1) that the flow rate at 1825 r/min and 1.49 kPa is Q_7 which is 524 m^3/min, not the desired flow rate of 481 m^3/min.

Let's see what happens if we use point 8 instead of point 7. Starting with equation 1 again:

$$N_{d3} = N_8 \frac{Q_d}{Q_8} = 1825\,\frac{r}{min}\,\frac{481\,\frac{m^3}{min}}{481\,\frac{m^3}{min}} = 1825.0\,\frac{r}{min}$$

We know from the fan curve data (fig. 1) this speed would give us the desired flow rate, but we also know that it will give us that flow rate at a pressure of 1.74 kPa, not the desired pressure of 1.49 kPa.

Using equation 2 with point 8 and the simplifying assumptions:

$$N_{d4} = N_8\sqrt{\frac{p_{td}}{p_{t8}}} = 1825\,\frac{r}{\min}\sqrt{\frac{1.49\,kPa}{1.74\,kPa}} = 1688.8\,\frac{r}{\min}$$

As a double-check, let's calculate the flow rate we should get with this speed, remembering that we want a flow rate of 481 m^3/min.

$$Q_d = Q_8\left(\frac{N_{d4}}{N_8}\right) = 481\,\frac{m^3}{min}\left(\frac{1688.8\,\frac{r}{min}}{1825\,\frac{r}{min}}\right) = 445\,\frac{m^3}{min}$$

We have now tried using two of the fan performance curve data points with two of the fan law equations, giving us four estimates of the speed at which we should operate the fan, and none of those speeds give us the combination of pressure and flow rate that we desire.

One method of arriving at a final solution is to use numerical techniques such as a linear optimization program to search for a solution. Two disadvantages of this approach are: the numerical technique may not always converge to a final solution, and typical numerical methods search for a solution within a specified error, which may need some adjustment from one situation to the next to obtain acceptable results. An analytical solution avoids these disadvantages.

Figure 2 shows the locations of the four points obtained from this analysis relative to the fan performance curve at 1825 r/min, and the desired operating point. We notice that the four points obtained from the analysis can be considered to form a diamond shape around the desired point. The two points obtained using point 7 form the bottom and right corners of the diamond, or the lower-right line of the diamond. The two points obtained using point 8 form the top and left corners of the diamond, or the upper-left line of the diamond. Considering those two lines, we infer that there must be one point on the fan performance curve, between data points 7 and 8, which would have a similar line that would pass through the desired operating point. This logic develops the concept that *the "basic point" is the one point on the fan performance curve that is related to the desired operating point by the fan laws*.

Analysis

The general shape of fan or pump performance curves (pressure as a function of flow rate) indicates an appropriate model for the curve is a second-order polynomial of the following form:

$$p = a_0 + a_1 Q + a_2 Q^2 \tag{5}$$

where

p = pressure,

Q = flow rate, and

a_0, a_1, and a_2 = the best fit, least-squares coefficients for the performance curve.

Given a few operating points (flow rate and pressure) for a particular fan or pump curve, or a graph of a fan or pump curve from which we can obtain these operating points, the coefficients for the best-fit, least-squares, second-order polynomial, and the coefficient of determination (r^2) for the curve can be determined. It is recommended that statistical parameters (such as the coefficient of determination, r^2) describing how well the second-order polynomial model fit the data be checked to ensure the model will give good results.

The requirement for the basic point is that it is the point on the fan or pump performance curve such that the affinity laws describe the relationships between the pressure, flow rate, and speed of the basic point (p_b, Q_b, N_b) and the corresponding pressure, flow rate, and speed of the desired operating point (p_d, Q_d, N_d) as follows:

$$Q_b = \frac{N_b}{N_d} Q_d \tag{6}$$

$$p_b = \left(\frac{N_b}{N_d}\right)^2 p_d \tag{7}$$

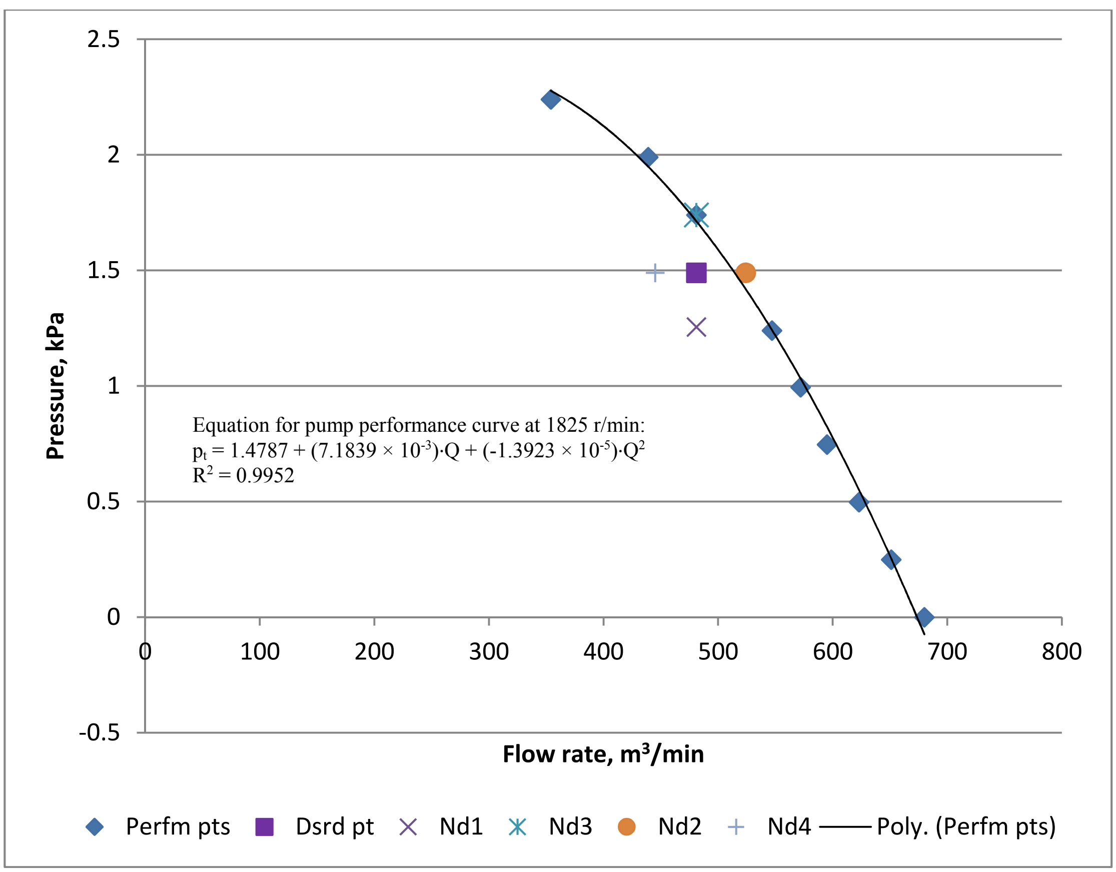

Figure 2. Fan performance curve at 1825 r/min, the desired operating point, and the four operating points determined using the fan laws with data points 7 and 8.

Since the basic point is on the fan or pump curve, we know: $p_b = a_0 + a_1Q_b + a_2Q_b^2$.

There are three ways to go through the analysis. One approach is to solve for the basic point flow rate (Q_b) first, and then solve for the desired operating point speed (N_d) and the basic point pressure (p_b). Another approach is to solve for the desired operating point speed (N_d) directly without first solving for the basic point flow rate (Q_b). The third approach is to solve for the basic point pressure (p_b) first, and then solve for the basic point flow rate (Q_b) and the desired operating point speed (N_d). The analysis for the third approach is more complicated than the first two and gives the same results, so we'll work through the first two approaches and not the third. As they are analytical solutions, all three approaches will give the same result.

Solution Approach 1

Using the first approach we solve for the basic point flow rate (Q_b) first, and then solve for the desired operating point speed (N_d) and the basic point pressure (p_b).

Substituting for p_b in the fan or pump curve equation we obtain: $\left(\frac{Q_b}{Q_d}\right)^2 p_d = a_0 + a_1Q_b + a_2Q_b^2$

Rearranging into the standard form for the quadratic equation in Q_b gives the following:

$$\left(a_2 - \frac{p_d}{Q_d^2}\right)Q_b^2 + a_1Q_b + a_0 = 0$$

Using the quadratic formula to solve for Q_b we obtain the following:

$$Q_b = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0\left(a_2 - \frac{p_d}{Q_d^2}\right)}}{2\left(a_2 - \frac{p_d}{Q_d^2}\right)}$$

With a solution for Q_b, we can calculate p_b and N_d:

$$p_b = \left(\frac{Q_b}{Q_d}\right)^2 p_d \text{ and } N_d = \frac{Q_d}{Q_b}N_b$$

Solution Approach 2

Using the second approach, we solve for the desired operating point speed (N_d) directly without first solving for the basic point pressure (p_b) and flowrate (Q_b).

Substituting for p_b and Q_b we obtain: $\left(\frac{N_b}{N_d}\right)^2 p_d = a_0 + a_1\left(\frac{N_b}{N_d}\right)Q_d + a_2\left(\frac{N_b}{N_d}\right)^2 Q_d^2$

Multiplying through by N_d^2, we obtain: $N_b^2 p_d = a_0N_d^2 + a_1N_bQ_dN_d + a_2N_b^2Q_d^2$.

Rearranging into the standard form for the quadratic equation in N_d gives the following:

$$a_0N_d^2 + a_1N_bQ_dN_d + N_b^2\left(a_2Q_d^2 - p_d\right) = 0$$

Using the quadratic formula to solve for N_d we obtain the following:

$$N_d = \frac{-a_1N_bQ_d \pm \sqrt{\left(a_1N_bQ_d\right)^2 - 4a_0N_b^2\left(a_2Q_d^2 - p_d\right)}}{2a_0}$$

Rearranging slightly and factoring N_b and Q_d out of the numerator terms gives the following form of the solution for the desired fan or pump speed:

$$N_d = \left(N_bQ_d\right)\frac{-a_1 \pm \sqrt{a_1^2 + 4a_0\left(\frac{p_d}{Q_d^2} - a_2\right)}}{2a_0}$$

Once the desired speed is known, the basic point can be determined as follows:

$$Q_b = \frac{N_b}{N_d}Q_d \text{ and } p_b = \left(\frac{N_b}{N_d}\right)^2 p_d$$

EXAMPLE 2

Let's go back and solve the problem from Example 1 using the analytical solutions of approaches 1 and 2.

The following data are obtained from the problem statement:

Q_d = 481 m^3/min

p_d = 1.49 kPa

N_b = 1825 r/min

The least-squares, best fit, second-order polynomial coefficients giving an r^2 value of 0.99524 were determined to be:

a_0 = 1.4787 kPa

a_1 = 7.1839 × 10^{-3} kPa·min/m^3

a_2 = -1.3923 × 10^{-5} kPa·min^2/m^6

The r^2 value was close to 1.0 indicating the second-order polynomial provides a nearly perfect fit to the fan performance curve data as can be seen in figures 1, 2, and 3.

Using Solution Approach 1

Calculating a value for the term $a_2 - p_d/Q_d^2$:

$$a_2 - \frac{p_d}{Q_d^2} = -1.3923\times10^{-5}\,\frac{kPa\cdot\min^2}{m^6} - \frac{1.49kPa}{\left(481\frac{m^3}{\min}\right)^2}$$

$$= -2.0363\times10^{-5}\,\frac{kPa\cdot\min^2}{m^6}$$

Calculating the two possible values for Q_b:

$$Q_{b,1} = \left[-7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3} + \sqrt{\left(7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3}\right)^2 - 4(1.4787\,kPa)\left(-2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}\right)}\right] \div \left[2\left(-2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}\right)\right] = -145.68\frac{m^3}{\min}$$

$$Q_{b,2} = \left[-7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3} - \sqrt{\left(7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3}\right)^2 - 4(1.4787\,kPa)\left(-2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}\right)}\right] \div \left[2\left(-2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}\right)\right] = 498.47\frac{m^3}{\min}$$

$Q_{b,1}$ = -145.68 m^3/min and $Q_{b,2}$ = 498.47 m^3/min, so the correct solution is Q_b = 498.5 m^3/min.

Using this to solve for p_b and N_d:

$$p_b = \left(\frac{Q_b}{Q_d}\right)^2 p_d = \left(\frac{498.5\frac{m^3}{\min}}{481\frac{m^3}{\min}}\right)^2 1.49\,kPa = 1.60\,kPa$$

$$N_d = \frac{Q_d}{Q_b}N_b = \frac{481\frac{m^3}{min}}{498.5\frac{m^3}{min}}1825\frac{r}{min} = 1761.0\frac{r}{min}$$

Using Solution Approach 2

$$\frac{p_d}{{Q_d}^2} - a_2 = -\left(a_2 - \frac{p_d}{{Q_d}^2}\right) = 2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}$$

Calculating the two possible values for N_d:

$$N_{d,1} = \left(1825\frac{r}{\min}\right)\left(481\frac{m^3}{\min}\right)\left[-7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3} + \sqrt{\left(7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3}\right)^2 + 4\,(1.4787\,kPa)\left(2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}\right)}\right] \div \left[2\cdot(1.4787\,kPa)\right]$$

$$N_{d,1} = 1761.0\ rpm$$

$$N_{d,2} = \left(1825\frac{r}{\min}\right)\left(481\frac{m^3}{\min}\right)\left[-7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3} - \sqrt{\left(7.1839\times10^{-3}\frac{kPa\cdot\min}{m^3}\right)^2 + 4\,(1.4787\,kPa)\left(2.0363\times10^{-5}\frac{kPa\cdot\min^2}{m^6}\right)}\right] \div \left[2\cdot(1.4787\,kPa)\right]$$

$$N_{d,2} = -6025.8\frac{r}{min}$$

Obviously the practical solution is the new fan speed of 1761.0 r/min and we can ignore the other solution.

Calculating the flow rate and pressure for the basic point (Q_b and P_b):

$$Q_b = \frac{N_b}{N_d}Q_d = \frac{1825\frac{r}{min}}{1761.0\frac{r}{min}}481\frac{m^3}{min} = 498.5\frac{m^3}{min}$$

$$p_b = \left(\frac{N_b}{N_d}\right)^2 p_d = \left(\frac{1825\frac{r}{\min}}{1761.0\frac{r}{\min}}\right)^2 1.49\,kPa = 1.60\,kPa$$

Whether solution approach 1 or 2 was used, the solutions for the basic point pressure and flow rate (1.60 kPa and 498.5 m^3/min) and fan speed to achieve the desired operating point (1761 r/min) were the same. The fan performance curve (at 1825 r/min), the desired operating point, and the basic point determined using both of the analytical solution approaches are depicted in figure 3.

CONCLUSION

Two analytical solution approaches were developed for determining the new fan (or pump) speed to achieve a desired operating point condition (flow rate and pressure combination). The "basic point" for these approaches is the point on the fan or pump performance curve such that the affinity laws describe the relationships between the pressure, flow rate and speed of the basic point (p_b, Q_b, N_b) and the corresponding pressure, flow rate, and speed of the desired operating point (p_d, Q_d, N_d).

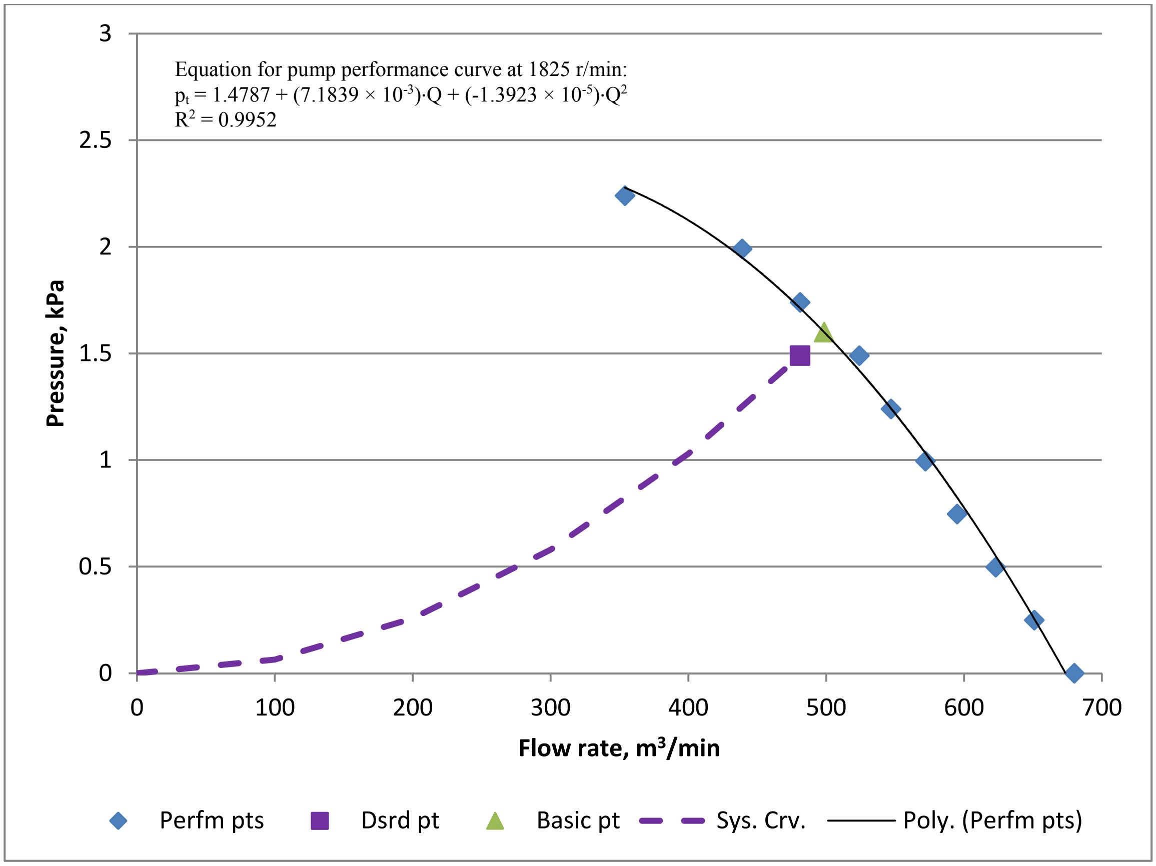

Figure 3. Desired operating point on the system curve, fan performance curve at 1825 r/min, and the basic point on the fan performance curve that corresponds with the desired operating point.

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