

University of Nebraska - Lincoln

DigitalCommons@University of Nebraska - Lincoln

Faculty Papers and Publications in Animal
Science

Animal Science Department

June 1993

Estimation of the Parameters Involved in a First-Order Autoregressive Process for Contemporary Groups

K. M. Wade
Cornell University

R. L. Quaas
Cornell University

L. Dale Van Vleck
University of Nebraska-Lincoln, dvan-vleck1@unl.edu

Follow this and additional works at: <https://digitalcommons.unl.edu/animalscifacpub>



Part of the [Animal Sciences Commons](#)

Wade, K. M.; Quaas, R. L.; and Van Vleck, L. Dale, "Estimation of the Parameters Involved in a First-Order Autoregressive Process for Contemporary Groups" (1993). *Faculty Papers and Publications in Animal Science*. 142.

<https://digitalcommons.unl.edu/animalscifacpub/142>

This Article is brought to you for free and open access by the Animal Science Department at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in Faculty Papers and Publications in Animal Science by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.

Estimation of the Parameters Involved in a First-Order Autoregressive Process for Contemporary Groups

K. M. WADE,¹ R. L. QUAAS,² and L. D. VAN VLECK³

Department of Animal Science
Cornell University
Ithaca, NY 14853

ABSTRACT

A methodology was developed for estimating the parameters involved in a first-order autoregressive process; these parameters comprise a variance component associated with the random effect, a correlation coefficient, ρ , and a residual variance. These parameters were estimated using REML with an expectation-maximization algorithm. For two single-trait analyses (milk and fat production being the dependent variable), the example chosen for the analyses was year-month—treated as random and following a first-order autoregressive process—within fixed herd. Initially, estimates failed to converge, possibly because of a time trend in the data, which was not accounted for by the model. After the random effect that follows the first-order autoregressive process was redefined as month within fixed herd-year, the parameters converged, and ρ was estimated as .8 for milk and fat yield. Results suggest that the estimation procedures may be useful for situations when a first-order autoregressive process seems appropriate.

(**Key words:** parameter estimation, autoregressive process, contemporary groups)

Abbreviation key: AR(1) = first-order autoregressive, CG = contemporary group, E = expectation operator.

INTRODUCTION

Use of autoregression for modeling of biological processes is often proposed [see, for example, (22)]. In particular, the first-order autoregressive [AR(1)] process represents an appealing method to model data that are associated with one another by means of a correlation coefficient within a defined time sequence.

We (22) previously showed how such a process can be incorporated into the mixed model equations with a minimum of extra computation and without restricting the model in other effects. An advantage of such a process is the need for only two extra parameters. The purpose of this paper is to derive methods for the estimation of those parameters, which are a necessity for any biological application. We propose using the same example that was used in a previous study (22), the treatment of contemporary groups (year-months) as random within herd.

Although the controversial subject of much discussion in the past, contemporary group (CG) has been traditionally treated as fixed in order to avoid possible bias due to selective use of sires across herds (5, 11, 12, 15, 17, 19, 20). This bias is evident when producers do not use a random sample of bulls available for breeding or when some animals are preferentially treated (2, 15). Treatment of CG as fixed will partially account for violations to usual assumptions about sampling.

However, Schaeffer (19) argued that CG is, in fact, a random effect and should be so treated to model biological aspects correctly. He further suggested (19) that heritability estimates may have been biased upward in the past by elimination of the additional variance component (added to the denominator of the heritability estimator) that would result from use of the more correct model. Also, Henderson (11, 12) showed that, for genetic evaluations in which herds are treated as random, the

Received August 17, 1992.

Accepted June 7, 1993.

¹Present address: Department of Animal Science, McGill University, Macdonald Campus, 21,111 Lakeshore Road, Ste. Anne de Bellevue, PQ, Canada H9X 3V9.

²Reprint requests.

³Present address: Agriculture Research Service, A218 Animal Science, University of Nebraska, Lincoln 68583.

prediction error variance of differences in two genetic estimates is smaller than when herds are treated as fixed.

Except for large herds, information may be lost when CG is treated as fixed (3), and this problem is especially significant for countries and evaluations using small herd sizes (6, 7, 17). Although most think that cows freshening in the same CG have environmental similarities and should be so grouped, Preisinger et al. (17) pointed out that, in populations with small herd sizes, record numbers are frequently insufficient for a CG when treated as fixed. Van Vleck (20) has demonstrated how the treatment of CG as random would increase the effective number of daughters for sire evaluation. Van Vleck (20) illustrates how, even for a CG containing only one sire, the information would not be discarded (as is presently the case for any evaluations when CG is treated as fixed), although the weighting on that information would be minimal. He also accepts that the danger of bias would be increased by treating CG as random. If this danger is ignored, then the proposal for treating CG as random and allowing for nonzero covariances is closely followed by the question of how to include it in a model. A previous study (22)

Although the specific application of AR(1) methodology to this area of CG is important, the main objective of this paper was to assess the feasibility of estimating the parameters involved in such a process, given that they can easily be incorporated into the "usual" mixed model equations (22), and to develop methods for their estimation.

MATERIALS AND METHODS

An AR(1) Process

The AR(1) is defined by Box and Jenkins (1) as

$$t_k = \begin{cases} \epsilon_k & \text{for } k = 1 \\ \rho t_{k-1} + \epsilon_k & \text{for } k > 1 \end{cases}$$

where t is the dependent variable (current value of the process), ρ is a correlation coefficient, ϵ is the residual, and k is a point in time. Following the notation developed in previous work (22), the (co)variance matrix associated with the vector \mathbf{t} (which allows for missing time periods) is shown in [1].

$$\mathbf{S} = \mathbf{H}\sigma_t^2 = \begin{bmatrix} 1 & \rho^{t_2-t_1} & \rho^{t_3-t_1} & \rho^{t_4-t_1} & \dots & \rho^{t_{n-1}-t_1} & \rho^{t_n-t_1} \\ \rho^{t_2-t_1} & 1 & \rho^{t_3-t_2} & \rho^{t_4-t_2} & \dots & \rho^{t_{n-1}-t_2} & \rho^{t_n-t_2} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \rho^{t_{n-1}-t_1} & \rho^{t_{n-1}-t_2} & \rho^{t_{n-1}-t_3} & \rho^{t_{n-1}-t_4} & \dots & 1 & \rho^{t_n-t_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{t_n-t_1} & \rho^{t_n-t_2} & \rho^{t_n-t_3} & \rho^{t_n-t_4} & \dots & \rho^{t_n-t_{n-1}} & 1 \end{bmatrix} \sigma_t^2, \quad [1]$$

where the superscripts t_k are consecutively available year-months, within a herd, arranged in ascending order from lowest (t_1) to highest (t_n).

The Model

The model used was $y_{ijk} = h_i + t_{j(i)} + e_{ijk}$, where y is the observation ijk (milk or fat production), h is the fixed effect of herd i , t is the year-month j nested within herd i and following an AR(1) process, and e is random residual ijk . In an effort to assess the estimation procedures required for an AR(1) process, no additional effects were considered. In matrix notation the model is

$$\mathbf{y} = \mathbf{W}\mathbf{h} + \mathbf{Q}\mathbf{t} + \mathbf{e},$$

where \mathbf{W} and \mathbf{Q} are model matrices that relate observations to \mathbf{h} and \mathbf{t} . Using E as the expectation operator,

$$E \begin{bmatrix} \mathbf{y} \\ \mathbf{t} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{W}\mathbf{h} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

and

$$\text{var} \begin{bmatrix} \mathbf{y} \\ \mathbf{t} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \mathbf{QS} & \mathbf{R} \\ \mathbf{SQ}' & \mathbf{S} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & \mathbf{R} \end{bmatrix},$$

where $V = QSQ' + R$, $S = H\sigma_t^2$, and $R = I\sigma_e^2$. The mixed model equations for this model are

$$\begin{bmatrix} W'W & W'Q \\ Q'W & Q'Q + H^{-1}\alpha \end{bmatrix} \begin{bmatrix} \hat{h} \\ \hat{t} \end{bmatrix} = \begin{bmatrix} W'y \\ Q'y \end{bmatrix};$$

$\alpha = \sigma_e^2/\sigma_t^2$. Because all herds are independent, the equations can be rearranged into a block-diagonal system of independent analyses [2] (one for each herd) and solved using methods developed by Wade and Quaas (22).

$$\begin{bmatrix} Q_i'Q_i + H_i^{-1}\alpha & Q_i'1_i \\ 1_i'Q_i & 1_i'1_i \end{bmatrix} \begin{bmatrix} \hat{t}_i \\ \hat{h}_i \end{bmatrix} = \begin{bmatrix} Q_i'y_i \\ 1_i'y_i \end{bmatrix} \text{ for } i = 1, \dots, s; \quad [2]$$

where s is the number of herds.

Parameter Estimation

The parameters were estimated using REML (16) with an expectation-maximization algorithm (9). The two-stage algorithm used in this study can be summarized. For the E step

$$\text{calculate } Q = E[\ln f(x|\phi) | K'y, \phi^{(p)}]. \quad [3]$$

For the M step: find $\phi^{(p+1)}$ in the parameter space that maximizes [3], where, using the notation of Dempster et al. (9), x are the complete data, $K'y$ are error contrasts of the incomplete data, and $\phi^{(p)}$ are estimates of the parameters at round p .

Finding [3] is equivalent to calculating $[\hat{x}\hat{x}' + \text{var}(\hat{x} - x)]$, where \hat{x} is BLUP($x|K'y, \phi^{(p)}$), and $\text{var}(\hat{x} - x)$ is the prediction error variance of \hat{x} . Estimation, therefore, reduces to the iterative process of finding $\phi^{(p+1)}$ from $\phi^{(p)}$ via an E step and an M step in which $\phi^{(p+1)}$ is found such that [3] is maximized. This cyclical procedure continues until $\phi^{(p+1)}$ is deemed to be equal to $\phi^{(p)}$. The Q function, referred to in [3], can now be written:

$$Q = \sum_{i=1}^s \{ \text{const} - \frac{1}{2} \ln |S_i^{-1}| - \frac{1}{2} \text{tr}(S_i^{-1}T_i) - \frac{1}{2} \ln |R_i^{-1}| - \frac{1}{2} \text{tr}(R_i^{-1}E_i) \}$$

for T_i defined as $[\hat{t}_i\hat{t}_i' + \text{var}(\hat{t}_i - t_i)]$, [4]

and E_i defined as $[\hat{e}_i\hat{e}_i' + \text{var}(\hat{e}_i - e_i)]$. [5]

Ignoring constants and substituting $I_1\sigma_e^{-2}$ for R_i^{-1} , Q simplifies to

$$\sum_{i=1}^s \left\{ -[q_i \ln \sigma_t^2 - \ln |H_i^{-1}|] - \frac{1}{\sigma_t^2} [\text{tr}(H_i^{-1}T_i)] - n_i \ln \sigma_e^2 - \frac{1}{\sigma_e^2} [\text{tr}(E_i)] \right\}, \quad [6]$$

for q_i and n_i , the number of year-months and the total number of observations, respectively, in a herd. The $p + 1$ estimates of the parameters are derived by taking partial derivatives of Q , [6], with respect to σ_e^2 , σ_t^2 , and ρ .

$$\frac{\partial Q}{\partial \sigma_e^2} = \frac{-N}{\sigma_e^2} + \frac{1}{\sigma_e^4} \sum_{i=1}^s \text{tr}(E_i)$$

for N = total number of observations in the data.

$$\text{By setting } \frac{\partial Q}{\partial \sigma_e^2} = 0, \quad \frac{N}{\sigma_e^2} = \frac{1}{\sigma_e^4} \sum_{i=1}^s \text{tr}(E_i).$$

Solving for σ_e^2 , the $p + 1$ estimate of σ_e^2 is calculated using estimates from round p ;

$$\sigma_e^{2(p+1)} = \left[\frac{\sum_{i=1}^s \text{tr}(E_i)}{N} \right]^{(p)} \quad [7]$$

Similarly,

$$\frac{\partial Q}{\partial \sigma_t^2} = \frac{-q}{\sigma_t^2} + \frac{1}{\sigma_t^4} \sum_{i=1}^s \text{tr}(H_i^{-1}T_i)$$

for q = total number of year-months in the data. Again, by equating to 0 and solving, the $p + 1$ estimate of σ_t^2 is

$$\sigma_t^{2(p+1)} = \left[\frac{\sum_{i=1}^s \text{tr}(\mathbf{H}_i^{-1} \mathbf{T}_i)}{q} \right]^{(p)} \quad [8]$$

Finally,

$$\begin{aligned} \frac{\partial Q}{\partial \rho} &= \sum_{i=1}^s \text{tr} \left(\mathbf{H}_i \frac{\partial \mathbf{H}_i^{-1}}{\partial \rho} \right) \\ &\quad - \frac{1}{\sigma_t^2} \sum_{i=1}^s \text{tr} \left(\frac{\partial \mathbf{H}_i^{-1}}{\partial \rho} \mathbf{T}_i \right). \end{aligned} \quad [9]$$

Because \mathbf{H}_i , as shown in [1], is different for each herd, setting [9] to 0 does not give a simple solution in terms of ρ ; in fact, the solution is highly nonlinear. Therefore, Fisher scoring was used to arrive at an appropriate estimate of ρ [see, for example, (18)]. This method involves the first partial derivative of Q with respect to ρ , [9], as well as the expectation of the second partial derivative of Q with respect to ρ , and is represented as

$$\rho^{(p+1)} = \rho^{(p)} - E \left[\left(\frac{\partial^2 Q}{\partial \rho^2} \right)^{-1} \right] \left(\frac{\partial Q}{\partial \rho} \right);$$

i.e., [9] divided by the expectation of [10].

$$\begin{aligned} \frac{\partial^2 Q}{\partial \rho^2} &= \sum_{i=1}^s \text{tr} \left(\frac{\partial \mathbf{H}_i}{\partial \rho} \frac{\partial \mathbf{H}_i^{-1}}{\partial \rho} + \mathbf{H}_i \frac{\partial^2 \mathbf{H}_i^{-1}}{\partial \rho^2} \right) \\ &\quad - \frac{1}{\sigma_t^2} \sum_{i=1}^s \text{tr} \left(\frac{\partial^2 \mathbf{H}_i^{-1}}{\partial \rho^2} \mathbf{T}_i \right). \end{aligned} \quad [10]$$

Fisher scoring reduces computations considerably because $E(\mathbf{T}_i) = \mathbf{H}_i \sigma_t^2$, and, therefore, $E[10]$ can be written as

$$\begin{aligned} \sum_{i=1}^s \text{tr} \left(\frac{\partial \mathbf{H}_i}{\partial \rho} \frac{\partial \mathbf{H}_i^{-1}}{\partial \rho} + \mathbf{H}_i \frac{\partial^2 \mathbf{H}_i^{-1}}{\partial \rho^2} \right) \\ - \frac{1}{\sigma_t^2} \sum_{i=1}^s \text{tr} \left(\mathbf{H}_i \frac{\partial^2 \mathbf{H}_i^{-1}}{\partial \rho^2} \right) \sigma_t^2, \end{aligned}$$

which simplifies to yield

$$E \left(\frac{\partial^2 Q}{\partial \rho^2} \right) = \sum_{i=1}^s \text{tr} \left(\frac{\partial \mathbf{H}_i}{\partial \rho} \frac{\partial \mathbf{H}_i^{-1}}{\partial \rho} \right). \quad [11]$$

Because elements of \mathbf{H}_i^{-1} are found simply as functions of differences in year-months (22), its first derivative with respect to ρ can be obtained by similar rules, shown in the Appendix. The derivative of \mathbf{H}_i with respect to ρ is easily computed via functions of time differences (year-months) and is also shown in the Appendix.

The only other components needed for the parameter estimation are those making up \mathbf{T}_i [4] and \mathbf{E}_i [5] as they pertain to [7], [8], and [9]. Quantities $\hat{\mathbf{t}}_i$ and $\hat{\mathbf{h}}_i$ are obtained following the methods described by Wade and Quaas (22). The inner product of $\hat{\mathbf{e}}'\hat{\mathbf{e}}$, used in [5], can then be obtained because $\hat{\mathbf{e}}_i = \mathbf{y}_i - \mathbf{Q}_i \hat{\mathbf{t}}_i - \mathbf{1}_i \hat{\mathbf{h}}_i = \mathbf{y}_i - \mathbf{M}_i \hat{\mathbf{s}}_i$, for \mathbf{M}_i defined as $[\mathbf{Q}_i; \mathbf{1}_i]$ and $\hat{\mathbf{s}}_i$ as $[\hat{\mathbf{t}}_i; \hat{\mathbf{h}}_i]'$. Therefore,

$$\begin{aligned} \hat{\mathbf{e}}_i' \hat{\mathbf{e}}_i &= \mathbf{y}_i' \mathbf{y}_i - \hat{\mathbf{s}}_i' \mathbf{M}_i' \mathbf{y}_i - \mathbf{y}_i' \mathbf{M}_i \hat{\mathbf{s}}_i \\ &\quad + \hat{\mathbf{s}}_i' (\mathbf{M}_i' \mathbf{M}_i) \hat{\mathbf{s}}_i = \mathbf{y}_i' \mathbf{y}_i - \hat{\mathbf{s}}_i' \mathbf{r}_i \\ &\quad - \hat{\mathbf{t}}_i' (\mathbf{H}_i^{-1} \alpha) \hat{\mathbf{t}}_i \end{aligned}$$

for \mathbf{r}_i defined as $[\mathbf{Q}_i' \mathbf{y}_i; \mathbf{1}_i' \mathbf{y}_i]$.

The matrix $[\text{var}(\hat{\mathbf{e}}_i - \mathbf{e}_i)]$ referred to in [5] is not required—only its trace—which is derived as

$$\begin{aligned} &\text{tr}[\text{var}(\hat{\mathbf{e}}_i - \mathbf{e}_i)] \\ &= \text{tr} \left\{ \text{var} \left[\mathbf{M}_i \begin{pmatrix} \mathbf{t}_i - \hat{\mathbf{t}}_i \\ \mathbf{h}_i - \hat{\mathbf{h}}_i \end{pmatrix} \right] \right\} \sigma_e^2 \\ &= \text{tr} (\mathbf{M}_i \mathbf{C}_i \mathbf{M}_i') \sigma_e^2. \end{aligned} \quad [12]$$

From [4], $\text{tr}[\text{var}(\hat{\mathbf{t}}_i - \mathbf{t}_i)]$ can be expressed as $\text{tr}[\mathbf{H}_i^{-1} \mathbf{C}_i^{11}]$, where \mathbf{C}_i^{11} is that part of \mathbf{C}_i corresponding to the year-months (\mathbf{t}_i) in that herd. This trace [13] is found by observing a similarity between its components and those of [12] [see (21) for more details] and simplifies to

$$[n_i - \text{tr}(\mathbf{M}_i \mathbf{C}_i \mathbf{M}_i')] \sigma_t^2. \quad [13]$$

where n_i is the number of year-months in herd i .

Although parameters would normally be estimated simultaneously, using a true expectation-maximization algorithm, they were estimated separately in this study; updates for

σ_e^2 and σ_t^2 were obtained in round $p + 1$, using estimates from round p . The $p + 1$ update for ρ was then estimated using the $p + 1$ estimate of σ_t^2 .

Field Data

The data used in these analyses were first lactation, 305-d, mature equivalent records for milk and fat from both grade and registered Holstein cows. The data were from Wisconsin for January 1970 through April 1985, and a summary of the final data is in Table 1. The average number of year-months per herd was approximately 32, which implies that data were observed, on average, every 6 mo within herd (the largest possible span would be 183 mo if data were only observed in January 1970 and April 1985).

Starting values for the residual variances were 1,357,213 kg² for milk and 1717 kg² for fat. These data were from a study (4) of the same data. Further, ρ was expected to be positive and was initially set at .5. A prior of 5 was assigned to σ_e^2/σ_t^2 ; therefore, the variance component associated with year-months was given a starting value of one-fifth the residual starting value.

RESULTS AND DISCUSSION

Parameter Estimation

Initial attempts at modeling year-month as random AR(1) within a herd failed; the estimate of ρ settled at 1.0, regardless of starting values, and the estimate of σ_t^2 then tended toward infinity. With each attempt, projection techniques discussed by Laird et al. (14) were used, but to no avail. The convergence criterion for ρ was satisfied because consecutive estimates were smaller than 10^{-6} , but only because of an imposed boundary of ± 1.0 . Because simulation work had been successful (23), there was no reason to suspect the methodology, and failure to account for a time trend in the field data was considered to be a possible reason for the lack of convergence. As a preliminary test, a trend was incorporated into the simulated data, and the same problem was observed as with the field data. Two options

were available at this stage; 1) remodel the data or 2) redefine the current model. The latter involved treatment of month as random AR(1) within herd-year, thereby accounting for some of the time trend of the fixed effect. This solution had advantages over the former. First, it required no change in programming (simply a redefinition of the effects), and, second, it allowed for the comparison with a similar study that looked at random month within herd-year (8). The maximum possible span was then 11 mo because the fixed effect was redefined as herd-year. This result would occur if a herd had freshenings in only January and December for any 1 yr. However, the average number of months with observations was 5.4.

This analysis was successful. Parameters for fat yield were the first to be estimated, and convergence was attained for all three parameters following 95 rounds of iteration, including three projections (14). The starting values already discussed were used in this case. For milk yield, the same starting values for the variance components as previously discussed were used, but the parameter ρ , obtained from the fat analysis, was used as the starting value here. Convergence was reached in 20 iterations with only one projection. Results of both analyses are in Table 2.

In general, the methodology developed for this particular model was satisfactory. Computation was not excessive, and no large-scale storage of elements was required. However, much of the efficiency of these methods stemmed from the simplicity of the model and because the system of equations could be reduced to essentially 52,325 (number of herd-years) independent analyses. This result would not have been the case had other effects, such as animal genetic effect, been included, leading to the addition of relationships and, therefore, a lack of block diagonality among herds. Al-

TABLE 1. The field data after edits.

Records, no.	534,017
Largest herd size, no.	741
Herds, no.	8724
Herd-years, no.	52,325
Herd-year-months, no.	279,961
Mean milk, kg	7476
Mean fat, kg	278

though addition of other effects would certainly increase the complexity of the parameter estimation—obtaining elements of the inverse of the coefficient matrix would be challenging—this procedure must only be done once. Having established the parameters, their inclusion in the mixed model methodology of routine evaluations is straightforward (22).

Application

With regard to the appropriateness of the application chosen in this study, final estimates of ρ (.8 for both milk and fat yield) are intuitively appealing, and both estimates of residual variance are in agreement with previous work (4). The only available comparison for estimates of σ_t^2 lies in work by Chauhan and Thompson (8). In attempting to account for covariances among cows freshening in the same CG, Chauhan (6) and Chauhan and Thompson (8) examined a notion similar to the one presented herein. Both of those studies (6, 8) used a "rolling months" model to examine month within herd-year for dairy sire evaluation. Chauhan and Thompson (8) estimated the covariances among records of cows freshening during the same month and between those of cows freshening at various intervals up to 11 mo apart. Based on their results, Chauhan (6) and Chauhan and Thompson (8) concluded that the covariances among months (within herd-year) seemed to follow a linear trend up to 5 mo apart and were thereafter small; these conclusions provided their justification for deciding on essentially a fifth-order moving average. Although their reason for considering only ≤ 5 -mo differences was because the 5-mo esti-

mate was the longest difference in months still exhibiting significant covariance, their estimates showed some evidence of covariance at an 8-mo gap, and their graphic representation of the data suggest that a quadratic fit to the covariances is also quite plausible, a fit that is entirely consistent for an AR(1) with a correlation of .8. Also, the two estimates of Chauhan and Thompson (8) of covariance for fat yield at less than 1 mo apart (the equivalent of σ_t^2 in this study) can only be approximated from their graph as 166 and 188 kg² for British Holstein data; the estimate of σ_t^2 for fat in this study was 197 kg². When month within herd-year was treated as random, following a fifth-order moving average, both studies (6, 8) found that accuracy of evaluation increased over the most simple model treating variation across CG as $I\sigma^2$ but concluded that the advantage of the rolling months model was probably not enough to outweigh the increased computing costs incurred. However, incorporation of an AR(1) has been demonstrated to result in a negligible increase in computing costs (22).

Other situations exist for which an AR(1) process seems applicable (22). For example, Kennedy and Schaeffer (13) proposed modeling cytoplasmic effects with such a process: compared with the example used in this study, herd-years would correspond to cytoplasmic sources or "cow families", and months within herd-year would correspond to generations within cytoplasmic source. The cytoplasmic variance would be estimated by σ_t^2 , and ρ would estimate the alteration rate of mitochondrial DNA. However, Kennedy and Schaeffer stress that relationships need to be included in the estimation process to avoid confounding between cytoplasmic and drift variances.

Harville (10) discussed the possible use of AR(1) models for analysis of time-periods within herd or cow effects from lactation to lactation and commented on the flexibility gained by this modeling procedure. Harville even proposed that higher order autoregressive processes be considered if AR(1) processes prove to be inadequate.

CONCLUSIONS

The methodology developed in this study for the estimation of parameters involved in an

TABLE 2. Parameter estimates for milk and fat yield with months treated as random within herd-year, using a first-order autoregressive structure.

Trait	Estimates of parameters ¹		
	$\hat{\rho}$	$\hat{\sigma}_t^2$	$\hat{\sigma}_e^2$
Fat, kg	.796	196.6	1739.6
Milk, kg	.793	133,711.5	1,367,048.9

¹ $\hat{\rho}$ is the estimate of the correlation between months within herd-year; $\hat{\sigma}_t^2$ is the estimate of the month variance; and $\hat{\sigma}_e^2$ is the estimate of the residual variance.

AR(1) process was effective and may be useful when an AR(1) seems appropriate. Although the model used in this study was simple, it could be helpful for the estimation of parameters in a more complex model. The estimate of the correlation (ρ) between months in the same herd-year of .8 seems realistic and can be argued to be in agreement with other research (8). When appropriate, the parameters in an AR(1) model, once estimated, can easily be included in routine mixed model methodology (22).

ACKNOWLEDGMENTS

The authors are grateful for the financial support given by the US-Spain Joint Committee for Scientific and Technological Cooperation and Eastern Artificial Insemination Cooperative, Inc. Computing support was provided by the Dairy Records Processing Laboratory, Ithaca, NY and the Cornell National Supercomputer Facility, which is funded in part by the National Science Foundation, New York State, and by IBM Corporation. Data were provided by the Animal Improvement Programs Laboratory, Agricultural Research Service, USDA, Beltsville, MD.

REFERENCES

- 1 Box, G.E.P., and G. M. Jenkins. 1976. *Time Series Analysis: Forecasting and Control*. Holden-Day Inc., Oakland, CA.
- 2 Brotherstone, S., and W. G. Hill. 1986. Heterogeneity of variance amongst herds for milk production. *Anim. Prod.* 42:297.
- 3 Brotherstone, S., W. G. Hill, and R. Thompson. 1989. Components of environmental variance of yields of cows in different lactations. *Anim. Prod.* 48:283.
- 4 Carabaño, M. J., K. M. Wade, and L. D. Van Vleck. 1990. Genotype by environment interactions for milk and fat production across regions of the United States. *J. Dairy Sci.* 73:173.
- 5 Chauhan, V.P.S. 1987. Dairy sire evaluation fitting some of the herd-year season effects as random. *Livest. Prod. Sci.* 16:117.
- 6 Chauhan, V.P.S. 1988. Evaluation of dairy sires in Swiss herds using a rolling months model. *J. Anim. Breed. Genet.* 105:294.
- 7 Chauhan, V.P.S., C. Hagger, F. Schmitz, and N. Kunzi. 1987. Comparison of herdclass and herd effect models in the estimation of breeding values of dairy sires in Switzerland. Page 196 in *Proc. 38th Annu. Mtg. Eur. Assoc. Anim. Prod.*, Lisbon, Portugal.
- 8 Chauhan, V.P.S., and R. Thompson. 1986. Dairy sire evaluation using a "rolling months" model. *J. Anim. Breed. Genet.* 103:321.
- 9 Dempster, A. P., N. M. Laird, and D. B. Rubin. 1977. Maximum likelihood from incomplete data via the EM algorithm. *J. R. Stat. Soc. Ser. B* 39:1.
- 10 Harville, D. A. 1979. Recursive estimation using mixed linear models with autoregressive random effects. Page 157 in *Variance Components and Animal Breeding. Proc. Conf. in Honor of C. R. Henderson*, Cornell Univ., Ithaca, NY.
- 11 Henderson, C. R. 1973. Sire evaluation and genetic trends. Page 10 in *Proc. Anim. Breeding Genet. Symp. in Honor of Dr. J. L. Lush*. Am. Soc. Anim. Sci., Am. Dairy Sci. Assoc., Champaign, IL.
- 12 Henderson, C. R. 1975. Best linear unbiased estimation and prediction under a selection model. *Biometrics* 31:423.
- 13 Kennedy, B. W., and L. R. Schaeffer. 1990. Reproductive technology and genetic evaluation. Page 507 in *Advances in Statistical Methods for Genetic Improvement of Livestock*. D. Gianola and K. Hammond ed. Springer-Verlag, Berlin, Germany.
- 14 Laird, N., N. Lange, and D. Stram. 1987. Maximum likelihood computations with repeated measures: application of the EM algorithm. *J. Am. Stat. Assoc.* 82: 97.
- 15 Meyer, K. 1987. Estimates of variance due to sire \times herd interactions and environmental covariances between paternal half-sibs for first lactation dairy production. *Livest. Prod. Sci.* 17:95.
- 16 Patterson, H. D., and R. Thompson. 1971. Recovery of inter-block information when block sizes are unequal. *Biometrika* 58:545.
- 17 Preisinger, R., J. Claus, and E. Kalm. 1986. Estimating variance components and breeding values in a dairy population with small herd size. *Proc. 3rd World Congr. Genet. Appl. Livest. Prod.*, Lincoln, NE XII: 409.
- 18 Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. 1986. *Numerical recipes: the art of scientific computing*. Cambridge Univ. Press, Cambridge, MA.
- 19 Schaeffer, L. R. 1987. Estimation of variance components under a selection model. *J. Dairy Sci.* 70:661.
- 20 Van Vleck, L. D. 1987. Contemporary groups for genetic evaluations. *J. Dairy Sci.* 70:2456.
- 21 Wade, K. M. 1990. Parameter estimation for mixed linear models with a first order autoregressive covariance structure. Ph.D. Diss., Cornell Univ., Ithaca, NY.
- 22 Wade, K. M., and R. L. Quaas. 1993. Solutions to a system of equations involving a first-order autoregressive process. *J. Dairy Sci.* 76:3026.
- 23 Wade, K. M., R. L. Quaas, and L. D. Van Vleck. 1990. Mixed linear models with an autoregressive error structure. *Proc. 4th World Congr. Genet. Appl. Livest. Prod.*, Edinburgh, Scotland, XIII:488.

APPENDIX

From previous work by Wade and Quaas (22), the diagonal and off-diagonal elements of

H_i^{-1} are simple functions of the time differences (year-months). Those elements are

$$\left[\frac{1}{1 - \rho^{2d_{k-1}}} + \frac{1}{1 - \rho^{2d_k}} - 1 \right]$$

and

$$- \left[\frac{\rho^{d_k}}{1 - \rho^{2d_k}} \right],$$

respectively, where the subscript k has range = 1, 2, ..., $n - 1$ (n is the number of year-months in that herd), and d_0 and d_n are equal to ∞ . The last requirement is necessary for the autoregressive process to be stationary and allows for the (1, 1) and (n , n) elements of H_i^{-1} to reduce to $\frac{1}{1 - \rho^{2d_1}}$ and $\frac{1}{1 - \rho^{2d_{n-1}}}$, respectively.

Once the elements of this matrix (H_i^{-1}) can be expressed as two distinct formulas, similar rules are easy to obtain for the elements of $(\partial H_i^{-1} / \partial \rho)$. This matrix is tridiagonal because

H_i^{-1} is tridiagonal and, allowing for the same restrictions on d_0 and d_n , the (k , k) and (k , $k + 1$) elements are

$$\left[\frac{2d_k - 1 \rho^{2(d_k - 1) - 1}}{(1 - \rho^{2d_{k-1}})^2} + \frac{2d_k \rho^{2(d_k) - 1}}{(1 - \rho^{2d_k})^2} \right]$$

$$\text{and } - \left[\frac{(d_k \rho^{2d_k} + d_k) \rho^{d_k - 1}}{(1 - \rho^{2d_k})^2} \right].$$

The matrix $(\partial H_i / \partial \rho)$ is not tridiagonal; however, its diagonal elements are all zero, and the element k of row j is

$$(t_k - t_j) \rho^{(t_k - t_j) - 1}$$

for $k = 1, n$ (where n is the order of H_i). However, because the only role of this matrix is in [11], where it is postmultiplied by $(\partial H_i^{-1} / \partial \rho)$, and because this matrix is tridiagonal, only the first off-diagonal of $(\partial H_i / \partial \rho)$ is needed for computations.