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IT is of interest to compare the occupation of phase space in electron, proton, and photon beams of good homogeneity, in relation to the Pauli exclusion principle. For particles exhibiting an exclusion principle, any increase in the homogeneity of the beam, any reduction in its angular divergence, represents a limitation on the maximum available current density. No such limit is evident for particles which do not obey an exclusion principle. The photon density encountered in laser beams, and the current density in electron and proton beams provide striking examples of this difference.

Consider a beam of particles traveling in the $+x$ direction, with angular divergence α and average x momentum component p_x. If the momentum uncertainties are represented in the usual way, we may write $\alpha = \Delta p_y/p_x = \Delta p_z/p_x$, so that

$$\Delta p_z = \Delta p_y = \alpha p_x. \tag{1}$$

A small volume of the beam of size $\Delta V_c = \Delta x \Delta y \Delta z$ will contain particles whose momentum components range through $p_x \pm \Delta p_x/2$, $\pm \Delta p_y/2$, and $\pm \Delta p_z/2$, and will therefore occupy a phase-space volume of

$$\Delta V_p = \Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z. \tag{2}$$

If the volume ΔV_c contains N particles, the density of particles in phase space will be

$$N/\Delta V_p = n/(\Delta p_x \Delta p_y \Delta p_z), \tag{3}$$

where n is the density of particles in the beam. Expressed in terms of the angular divergence α, this equation becomes

$$N/\Delta V_p = n/(p_x^2 \Delta p_x \alpha^2). \tag{4}$$

If, in a volume h^3 of phase space we find, on the average, g particles, the density of particles in phase-space is g/h^3. Inserting this notation into Eq. (4) we find

$$n = g(p_x^2 \Delta p_x \alpha^2/h^3). \tag{5}$$

Thus we see that the density of particles in a

beam depends on the uncertainty in the momentum of the particles, the square of the angular divergence of the beam, the square of the average momentum of the particles, and the average density of the particles in phase space.

We may re-express Eq. (5) as the current density in a beam of nonrelativistic electrons which have been accelerated through a potential V. We then have $p_x=(2mVe)^{\frac{1}{2}}$, and the current density in the beam is

$$j=nev=g(2me^3\alpha^2 V\Delta V/h^3). \tag{6}$$

Choosing typical values for a cathode ray beam as $V=20$ kV$\pm$0.5 V, and $\alpha=1$ mrad, the limiting current density is found by setting $g=2$, for the two spin states permitted at each point in phase space. We find this value to be $j_{\max}\approx 10^8$ A/cm^2. Since electron beam currents are nominally of the order of milliamperes per square millimeter, the electron beams of cathode ray tubes are normally not phase-space limited by large factors.

The pre-injector to the Argonne Zero Gradient Synchrotron is a Crockroft–Walton set which provides a proton-beam current[1] of 150 mA at 750 kV$\pm$100 V, at an angular divergence of 10 mrad, within a target of 1 cm diam. From these values we find $g\approx 5\times 10^{-18}$. Since the maximum allowable value of g is 2, the available phase space is rather sparsely occupied.

Let us now consider the occupation of photon beams.

The Planck radiation formula for the frequency distribution of radiant energy within a blackbody is a product of four factors: the number of possible polarizations, the number of points of phase space per unit frequency interval, the energy of a photon, and the probability of occupation of a phase-space point. Thus, we have

$$\rho_\nu d\nu=2\times\frac{4\pi\nu^2 d\nu}{c^2}\times h\nu\times\frac{1}{e^{h\nu/kT}-1}. \tag{7}$$

Of these factors the product of the first and the last give the factor g which we have used above to represent the average number of particles occupying a phase-space point. Thus, for blackbody radiation we have

$$g=2/(e^{h\nu/kT}-1). \tag{8}$$

At the temperature of the solar photosphere (approximately 6000°K) we find the phase-space occupation of a blackbody to be $g=0.04$, at $\lambda=6000$ Å. The sun's visible light does not provide a phase-space population which approaches the restrictions of the (inapplicable) Pauli principle.

The energy flux in a photon beam may be obtained from Eq. (5) by noting that $p_x=h/\lambda=h\sigma$, so that

$$S=nch\nu=g(\Delta\sigma hc^2\alpha^2/\lambda^3). \tag{9}$$

For a typical gas-laser beam[2] the power output may be taken as 4 mW over a 1 cm diam circle at $\lambda=1.153\ \mu$, with a beam divergence of 32 sec and a linebreadth of 7 kc. With these values we find $g=2\times 10^{12}$, as compared to a maximum value of 2 for fermions. If photons were fermions a laser with the same beam divergence and linewidth would have a maximum power output of 4×10^{-15} W. To produce the same phase-space occupancy at this wavelength a blackbody at 10^{16}°K is required, a temperature not even encountered in stellar interiors.

The output of the gas laser may be taken as direct experimental evidence for the boson character of photons; photons clearly do not obey an exclusion principle.

[1] Lee Teng, private communication.

[2] D. R. Herriott, J. Opt. Soc. Am. 52, 31 (1962).