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Iterative Nonlinear Pulsations in Massive Stars. I. The Iterative Approach

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Summary. We consider "soft" pulsations in a massive main-sequence star, and attempt to investigate the regime in which the motion is nonlinear but not strongly so. The star is assumed to begin its finite amplitude pulsation in the linear fundamental mode. The adiabatic equation of motion is expanded (schematically) to third order in the radial amplitude, and a solution sought in the form of a Fourier series over an interval which turns out not to deviate much from the fundamental period of pulsation. Arguments based on the size of successive terms are used to isolate a set of five equations, the iterative solution of which should provide a complete and consistent description of weakly nonlinear adiabatic pulsations to the order required. Finally, it is shown that the existence of resonances does not pose any *a priori* objection to the iterative technique.

Key words: nonlinear pulsations – massive stars – iterative technique

I. Introduction

This is the first of a series of articles, the purpose of which is to examine in detail the transition from the linear regime to the nonlinear regime in "soft" stellar pulsations. In this connection we shall treat adiabatic radial pulsations in a massive main-sequence star and attempt to study the transition to nonlinearity via an extension of the familiar stability integral of Eddington (Ledoux, 1958) to next highest order.

The approach we shall use toward nonlinear oscillations is also due originally to Eddington (1919) who expanded the adiabatic pulsation equation to second order introducing a new amplitude proportional to $\cos 2\sigma t$ on the simple argument that the presence of the square of the linear term (i.e. $\cos^2 \sigma t$) necessitated it. In the present article we shall treat the adiabatic equation of motion to third order (since this is the order necessary for our eventual task of extending the stability integral) and shall argue that, given a few straightforward conditions, the iterative scheme of Eddington should be consistently correct to the order required.

II. The Iterative Scheme

We begin with the stellar equations of motion and of mass conservation. With the substitution

$$r = a(1+\eta),$$

where a is the equilibrium radial extension variable and η is a small correction, we may derive the form

$$\varrho_0 \frac{ad^2\eta}{dt^2}(1-2\eta+3\eta^2) = L(\eta) - Q(\eta) + R(\eta), \tag{1}$$

valid to third order in η and its derivatives. Here $\varrho_0(a)$ is the equilibrium density, and $L(\eta)$, $Q(\eta)$ and $R(\eta)$ are spatial operators containing terms of first, second and third order, respectively.

For adiabatic motion in an ideal gas the expressions $L(\eta)$ and $Q(\eta)$ are given by Rosseland (1949). $R(\eta)$ will be evaluated by us in a subsequent paper. For our present purposes, however, explicit forms for these functions are not necessary; it will suffice merely to write them schematically as

$$L(\eta) = b_i(a)\eta^{(i)}$$
$$Q(\eta) = b_{ij}(a)\eta^{(i)}\eta^{(j)}$$
$$R(\eta) = b_{ijk}(a)\eta^{(i)}\eta^{(j)}\eta^{(k)}$$

where the summation convention is here and henceforth in force unless otherwise noted, and

$$\eta^{(i)} \equiv \frac{d^i\eta}{da^i}; i, j, k = 0, 1, 2.$$

Now the linear equation

$$\varrho_0 a \frac{d^2\eta}{dt^2} = L(\eta), \tag{2}$$

has the harmonic solutions (no summation convention)

$$\eta_k = \lambda \xi_k \cos \sigma_k t\,, \qquad k = 0, 1, 2, \ldots$$

where each σ_k is the angular frequency of a normal mode and $\lambda \xi_k$ the corresponding eigenfunction, the ξ_k being orthogonal with respect to $\varrho_0 a^4$ (Ledoux and Walraven, 1958) and normalized such that

$$\int_0^R \varrho_0 a^4 \xi_i \xi_j da = \delta_{ij}\,.$$

Here λ is a small scaling constant, introduced for convenience is what follows.

Let us now seek a solution to Eq. (1) in the form of a Fourier series (see also, in this connection, Gurm, 1963 and references therein)

$$\eta = \sum_0^\infty B_n \cos n\nu t + \sum_1^\infty C_m \sin m\nu t \tag{3}$$

subject to the conditions

$$B_1 = \lambda \xi_0 + O(\lambda^3);\ |B_{n'}|,\ |C_m| \lesssim \lambda^2\ (n' \neq 1)\,. \tag{4}$$

The physical meaning of these conditions is simply that, so far as the linear regime is concerned, one and only one of the normal modes subsists and grows, the others being damped out rapidly. We have taken the subsisting mode to be the fundamental. A discussion of this assumption, and of its consistency with the results obtained, will be undertaken in a later section.

Substituting Eq. (3) into Eq. (1) and, by virtue of the conditions given in Eq. (4), throwing out all terms of order higher than B_1^3, $B_1 B_{n'}$, $B_1 C_m$, we obtain to third order in λ

$$\begin{aligned}
&-\varrho_0 a\nu^2 \{n^2 B_n \cos n\nu t + m^2 C_m \sin m\nu t - 2[B_1^2 \cos^2 \nu t \\
&+ (1+n'^2) B_1 B_{n'} \cos \nu t \cos n'\nu t \\
&+ (1+m^2) B_1 C_m \cos \nu t \sin m\nu t] \\
&+ 3B_1^3 \cos^3 \nu t\} = L(B_n) \cos n\nu t + L(C_m) \sin m\nu t \\
&- Q(B_1) \cos^2 \nu t - \mathcal{Q}(B_1, B_{n'}) \cos \nu t \cos n'\nu t \\
&- \mathcal{Q}(B_1, C_m) \cos \nu t \sin m\nu t + R(B_1) \cos^3 \nu t\,,
\end{aligned}$$

where $\mathcal{Q}(B_1, B_k)$ has the form

$$\mathcal{Q}(B_1, B_k) = c_{ij} B_1^{(i)} B_k^{(j)}\,.$$

Finally, with the aid of some simple trigonometric relations among products of sines and cosines, we obtain

$$\begin{aligned}
&-\varrho_0 a\nu^2 \{n^2 B_n \cos n\nu t + m^2 C_m \sin m\nu t - B_1^2 (1 + \cos 2\nu t) \\
&- (1+n'^2) B_1 B_{n'} [\cos(n'+1)\nu t + \cos(n'-1)\nu t] \\
&- (1+m^2) B_1 C_m [\sin(m+1)\nu t + \sin(m-1)\nu t] \\
&+ \tfrac{3}{4} B_1^3 [\cos 3\nu t + 3 \cos \nu t]\} = L(B_n) \cos n\nu t \\
&+ L(C_m) \sin m\nu t - \tfrac{1}{2} Q(B_1)[1 + \cos 2\nu t] \\
&- \tfrac{1}{2} \mathcal{Q}(B_1, B_{n'})[\cos(n'+1)\nu t \\
&+ \cos(n'-1)\nu t] - \tfrac{1}{2} \mathcal{Q}(B_1, C_m)[\sin(m+1)\nu t \\
&+ \sin(m-1)\nu t] \\
&+ \tfrac{1}{4} R(B_1)[\cos 3\nu t + 3 \cos \nu t]\,.
\end{aligned}$$

We now equate coefficients of Fourier terms:

$\cos \nu t$

$$\begin{aligned}
L(B_1) + \varrho_0 a\nu^2 B_1 &= \varrho_0 a\nu^2 [2B_1 B_0 + 5B_1 B_2 - \tfrac{9}{4} B_1^3] \\
&\quad + \tfrac{1}{2}[\mathcal{Q}(B_1, B_0) + \mathcal{Q}(B_1, B_2)] - \tfrac{3}{4} R(B_1);
\end{aligned} \tag{5}$$

time independent

$$L(B_0) = \varrho_0 a\nu^2 B_1^2 + \tfrac{1}{2} Q(B_1) \tag{6}$$

$\cos 2\nu t$

$$\begin{aligned}
&-\varrho_0 a\nu^2 [4B_2 - B_1^2 - 10 B_1 B_3] \\
&= L(B_2) - \tfrac{1}{2} Q(B_1) - \tfrac{1}{2} \mathcal{Q}(B_1, B_3);
\end{aligned} \tag{7}$$

$\cos 3\nu t$

$$\begin{aligned}
&-\varrho_0 a\nu^2 [9B_3 - 5B_1 B_2 - 17 B_1 B_4 + \tfrac{3}{4} B_1^3] \\
&= L(B_3) - \tfrac{1}{2} \mathcal{Q}(B_1, B_2) - \tfrac{1}{2} \mathcal{Q}(B_1, B_4) + \tfrac{1}{4} R(B_1);
\end{aligned} \tag{8}$$

$\cos i\nu t\,(i > 3)$ *(no summation convention)*

$$\begin{aligned}
&L(B_i) + \varrho_0 a\nu^2 i^2 B_i \\
&= \varrho_0 a\nu^2 B_1 \{[1 + (i-1)^2] B_{i-1} + [1 + (i+1)^2] B_{i+1}\} \\
&+ \tfrac{1}{2} [\mathcal{Q}(B_1, B_{i-1}) + \mathcal{Q}(B_1, B_{i+1})]\,;
\end{aligned} \tag{9}$$

$\sin m\nu t$ *(no summation convention)*

$$\begin{aligned}
&L(C_m) + \varrho_0 a\nu^2 m^2 C_m \\
&= \varrho_0 a\nu^2 B_1 \{[1 + (m-1)^2] C_{m-1} + [1 + (m+1)^2] C_{m+1}\} \\
&+ \tfrac{1}{2} [\mathcal{Q}(B_1, C_{m-1}) + \mathcal{Q}(B_1, C_{m+1})]\,.
\end{aligned} \tag{10}$$

Now consider Eq. (7). The terms in $B_1 B_3$ are at least one order higher than B_1^2; we thus ignore the former for a moment obtaining

$$4\varrho_0 a\nu^2 B_2 + L(B_2) = \varrho_0 a\nu^2 B_1^2 + \tfrac{1}{2} Q(B_1)\,. \tag{11}$$

We shall show later that $\nu^2 = \sigma_0^2 + O(\lambda^2)$. Anticipating this result, and ignoring terms of order greater than λ^3, we get

$$4\varrho_0 \sigma_0^2 B_2 + L(B_2) = \varrho_0 a\sigma_0^2 \lambda^2 \xi_0^2 + \tfrac{1}{2} \lambda^2 Q(\xi_0)\,. \tag{12}$$

Thus

$$B_2 \approx \lambda^2\,.$$

Furthermore, Eq. (6) becomes

$$L(B_0) = \varrho_0 a\sigma_0^2 \lambda^2 \xi_0^2 + \tfrac{1}{2} \lambda^2 Q(\xi_0)\,, \tag{13}$$

and hence

$$B_0 \approx \lambda^2\,.$$

Let us turn to Eq. (8). The terms in $B_1 B_4$ can be at best $\approx B_1^3$; thus

$$B_3 \approx B_1^3\,,$$

justifying our use of the equality sign in Eq. (11).

We might note that Eqs. (12) and (13) are formally just Eqs. (7.11) and 7.12) of Rosseland (1949).

Now examine Eq. (5). Its RHS is formally two orders smaller than its LHS. If we neglect the RHS, we just get the linear solution

$$B_1 = \lambda \xi_0 \cos \sigma_0 t .$$

On the other hand, the terms on the RHS are of order λ^3 and thus should have a role to play. To see this let us set

$$B_1 = \lambda \xi_0 + D_1$$
$$v^2 = \sigma_0^2 + \alpha$$

where $D_1 \approx \lambda^3$ and α is a small correction $\lesssim \lambda$. Using these expressions in Eq. (5), we obtain

$$\begin{aligned} &L(D_1) + \varrho_0 a \alpha \lambda \xi_0 + \varrho_0 a \sigma_0^2 D_1 \\ &= \varrho_0 a \sigma_0^2 \lambda [2 \xi_0 B_0 + 5 \xi_0 B_2 - \tfrac{9}{4} \lambda^2 \xi_0^3] \\ &+ \tfrac{1}{2} \lambda [\mathcal{Q}(\xi_0, B_0) + \mathcal{Q}(\xi_0, B_2)] - \tfrac{3}{4} \lambda^3 R(\xi_0) . \end{aligned} \tag{14}$$

It is now easy to show that $\alpha \approx \lambda^2$. Once we have solved the second-order Eqs. (12) and (13), the RHS of Eq. (14) is known as a function of a and thus

$$L(D_1) + \varrho_0 a \sigma_0^2 D_1 + \varrho_0 a \alpha \lambda \xi_0 = \lambda^3 f(a) .$$

Expanding D_1 in terms of the complete set ξ_j

$$D_1 = \sum_1^\infty d_j \xi_j ,$$

we obtain

$$\varrho_0 a d_j (\sigma_0^2 - \sigma_j^2) \xi_j + \varrho_0 a \alpha \lambda \xi_0 = \lambda^3 f(a) .$$

Multiplying by $a^3 \xi_0$ and integrating, we find

$$\alpha = \lambda^2 \int_0^R a^3 f(a) \xi_0 da ,$$

while the $d_j (j \neq 0)$ are given by

$$d_j = \frac{\lambda^3}{(\sigma_0^2 - \sigma_j^2)} \int_0^R a^3 f(a) \xi_j da .$$

Since the normal modes are nondegenerate the d_j are all finite or zero, and, with α known, Eq. (14) should have a solution $D_1(a)$.

Let us turn to Eq. (9). First note that if the RHS were to vanish, B_i would either be (redundantly) identical to the linear fundamental eigenfunction $\lambda \xi_0$ or else identical to an eigenfunction corresponding to one of the other normal modes of the linear system, the latter case being contrary to the conditions we have put on our problem. For nonzero RHS, we shall consider Eq. (9) for some large $i = I$, and shall assume for the moment that

$$|B_{i+1}| < |B_{i-1}| . \tag{15}$$

This implies that $|\mathcal{Q}(B_1, B_{i+1})| < |\mathcal{Q}(B_1, B_{i-1})|$, and ignoring the $(i+1)$-terms compared with the $(i-1)$-terms, we obtain from Eq. (9)

$$|B_i| \lesssim |B_1 B_{i-1}| .$$

A moment's reflection satisfies one that this is in fact a relation which holds for any i such that $3 < i \leqq I$, and since it has already been established that $B_3 \approx \lambda^3$, we have

$$B_i \approx \begin{cases} \lambda^i \text{ for } i > 0 \\ \lambda^2 \text{ for } i = 0 \end{cases} , \tag{16}$$

provided that the inequality (15) holds as $I \to \infty$. But this inequality in the limit of large i is a necessary condition for the convergence of our Fourier series [Eq. (3)] in the first place.

In a similar manner, beginning with Eq. (10) for $m = 1$, it may be established that $C_m \approx \lambda C_{m+1}$, and thus that we must have

$$C_m \equiv 0 \text{ for all } m , \tag{17}$$

in order that our series converge.

Thus provided that the Fourier series converges, our analysis goes through as above and the orders of all terms are given by Eqs. (16) and (17). In that event, it is clear that to third order the only amplitudes that need to be considered are B_0, B_1, B_2, and B_3. Furthermore, we may now write Eq. (8) as

$$\begin{aligned} &L(B_3) + 9 \varrho_0 a \sigma_0^2 B_3 \\ &= \varrho_0 a \sigma_0^2 \lambda \xi_0 [5 B_2 - \tfrac{3}{4} \lambda^2 \xi_0^2] \\ &+ \tfrac{1}{2} [\mathcal{Q}(B_1, B_2) - \tfrac{1}{2} \lambda^3 R(\xi_0)] , \end{aligned} \tag{18}$$

completing a set of equations [Eqs. (2), (12)–(14) and (18)] which may be solved successively to calculate pulsational amplitudes to third-order.

III. Consistency of the Approach

Let us consider infinitesimal pulsations in a massive main-sequence star, and assume random perturbations such that some, or even all, of the normal modes (ξ_j, σ_j) are excited. If the star exceeds a certain critical mass (Schwarzschild and Harm, 1959; Stothers and Simon, 1970), but is not supermassive, it is well known that nuclear reactions in the core will energize the fundamental mode, while all of the higher modes will be strongly damped (e.g. Simon and Stothers, 1969). In that event, as the pulsations approach finite amplitude, the star should be oscillating strictly in its fundamental mode $\lambda \xi_0 \cos \sigma_0 t$. This is our starting point and it provides the rationale behind the conditions given in Eq. (4).

As the pulsation enters the nonlinear regime, but is not yet strongly nonlinear, we anticipate that the period should not differ much from the fundamental period $2\pi/\sigma_0$ (e.g. Ledoux and Walraven, 1958 and references therein) and that the amplitudes $\eta(a, t)$ should be

smooth and well-behaved. For these reasons, it seems quite justified to expand $\eta(a, t)$ in a Fourier series over an interval defined by an angular frequency $\nu \simeq \sigma_0$, and further to assume that the function is indeed piecewise very smooth (e.g., Kaplan, 1952) over this interval so that the series will converge.
Given these assumptions, we have seen above that a hierarchy of terms falls out neatly with orders as specified in Eqs. (16) and (17), that the period change emerges in a natural way, and that no noticeable inconsistencies arise. We thus find it reasonable that, with one notable exception, we should be able to choose a λ large enough so that the nonlinear terms have some effect, yet small enough such that both the order of our approximation and the smooth behavior of the pulsation are preserved. Exactly what values are appropriate for λ will be discussed in detail in future articles in connection with specific stellar models. The notable exception mentioned above is that of resonant pulsations, and we turn our attention to this question now.

IV. The Effect of Resonances

We have every reason to expect that whenever a resonance exists between the fundamental and another of the normal modes, i.e., for $\sigma_j \simeq \text{integer} \times \sigma_0$, the former, as it grows, will pick up the Fourier term corresponding to the resonant overtone, resulting in a possible violation of the conditions given in Eq. (4). We should not be very concerned about a resonant coupling involving one of the lower modes, since, for a given model, we can calculate these modes and, in the event of a resonance, alter the model parameters somewhat to get rid of it. However, the number of modes is infinite, approximate equalities of the form given above are inevitable somewhere down the line, and it vitiates our entire approach if we cannot find some means of showing that large resonant effects will, in general, not appear in the higher terms of the Fourier expansion.
The key to this question lies in the fall-off of the Fourier amplitudes, enumerated in Eq. (16). It was recognized by Ledoux and Walraven (1958) that due to this fall-off, the higher order terms become harder and harder to excite resonantly, and it is just this fact which offers an excape from the quandary posed above.

Simon and Sastri (1972; henceforth SS) have studied resonance effects in a number of polytropic models. Following their discussion, let us define the difference

$$Z_{kj} = \frac{R^3}{GM}(k^2\sigma_0^2 - \sigma_j^2) \qquad (k \text{ an integer})$$

between k^2 times the square of the dimensionless angular frequency of the fundamental mode, and the square of the j-th normal mode. Asymptotically, for small Z_{kj}, we expect the coefficient of the k-th Fourier term to blow up like Z_{kj}^{-1} (e.g., Rosseland, 1949). Furthermore, it is shown in SS that asymptotically for large j

$$\frac{\sigma_j}{\sigma_0} = a_1 j + a_2$$

where j is, of course, a positive integer, and a_1 and a_2 are positive irrational numbers of order unity.
Now we shall have a resonance whenever Z_{kj} is very small, i.e., whenever

$$a_1 j + a_2 = k$$

is satisfied for an approximate integer pair (j, k). Since a_1 and a_2 are of order unity, $k \approx j$. For our purposes, it shall suffice to consider a_1 and a_2 as decimal fractions, each with m places. In the absence of an indication that nature either favors or frowns upon resonant pulsations, we shall assume that each pair (a_1, a_2) is equally likely to occur in a real star. Then, with a given a_1, it is clear that for each integer j, there is one and only one a_2 out of 10^m possible a_2's such that k will be an integer to m decimal places. Now if j is an m-digit integer, there are 10^m j's and thus a probability $P \lesssim 1$ that a resonance occurs to the accuracy specified; if j has $m-1$ digits, $P \lesssim 1/10$; or, generally, $P \lesssim 1/10^{m-n}$, where n $(1 \leqq n \leqq m)$ is the number of digits in the integer j.
As an example of the above consider $a_1 = 0.8723$. Then for $2 \leqq j \leqq 9$ there are 8 favorable values of a_2 (e.g., $a_2 = 0.8939$ for $j = 7$) out of 10^4 possible values; for $10 \leqq j \leqq 99$, there are 10^2 favorable a_2's; for $100 \leqq j \leqq 999$, 10^3 favorable a_2's; etc.
We are now ready to make some estimate of the probability that a noticeable resonance occurs in a high Fourier term, say the k-th term. In the event of a resonance, we can crite

$$B_k \approx \frac{\lambda^k}{Z_{kj}} (\lambda < 1)\,.$$

Since a term must be $\gtrsim \lambda^3$ to be considered in our treatment, we require

$$|Z_{kj}| = \frac{R^3}{GM}(k\sigma_0 + \sigma_j)\,\sigma_0\,|k - a_1 j - a_2| \lesssim \lambda^{k-3}\,. \qquad (19)$$

A single example will now suffice to show improbable would be the appearance of a noticeable resonance in a high Fourier term. We first note that for nearly any reasonable stellar model the factor $R^3(k\sigma_0 + \sigma_j)\,\sigma_0/GM$ will always exceed unity (see, e.g., SS). Now let us choose a modest (i.e., fairly large) $\lambda = 1/4$ and ask that the resonance occur for $10 \leqq j, k \leqq 99$. For the best case, $k = 10$, an effective resonance requires that the absolute bracket in Eq. (19) be $< 10^{-4}$; the probability of this happening is 10^{-2}. On the other hand, for the worst case, $k = 99$, we require the absolute bracket $\approx 10^{-57}$, yielding, of course, a probability that is vanishingly

small. As k gets even larger, the situation clearly becomes worse.

It should be noted here that the picture is not quite as we have portrayed it in that actually a_1 and a_2 are coupled for any given model and thus cannot be chosen independently. However, in view of the fact, mentioned above, that the work of SS shows no tendency for this coupling to favor resonances, the conclusion stands that an effective resonance beyond the first few modes is highly improbable.

Finally it may be remarked that, in a real physical pulsation, the presence of nonconservative terms will tend to mitigate resonant effects on the amplitudes, while perhaps introducing interesting phase shifts between the modes involved.

V. Conclusion

It has been shown here that, given the straightforward and realistic physical assumptions outlined in Section III, the iterative approach of Eddington provides a logical and consistent method for treating nonlinear pulsations, and that a solution, complete to third-order, may be found from the set of Eqs. (2), (12–14), (18). Furthermore, it has been demonstrated that the possible existence of resonances poses no *a priori* objection to the iterative technique, but only affects it *a posteriori* when low-order resonant couplings happen to appear in specific models.

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