3-23-1972

Iterative Nonlinear Pulsations in Massive Stars. II. Terms up to Second Order

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Astron. & Astrophys. 21, 51–55 (1972)

Iterative Nonlinear Pulsations in Massive Stars. II. Terms up to Second Order

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Received March 23, 1972

Summary. An iterative technique is used to calculate radial adiabatic pulsational amplitudes up to second order in a very massive main-sequence star. The second-order terms are displayed, and discussed in detail. A possible connection between the qualitative structure of these terms and the absence of observed massive main-sequence pulsators is suggested. Finally, we discuss the resonance properties of the present model and consider their effect on the iterative calculation.

Key words: nonlinear pulsations – massive stars – iterative technique

I. Introduction

With this article, we begin the chain of calculations whose eventual aim is the physical analysis of small amplitude, nonlinear *adiabatic* pulsations by means of the extension of the standard pulsational stability integral (e.g., Ledoux, 1958) to next highest order. The line of attack we have chosen for evaluation of the nonlinear terms is an extension of an iterative method due to Eddington (Rosseland, 1949) in which the radial amplitude is expanded in a Fourier series whose various coefficients represent nonlinear terms of differing orders. The iterative technique consists in the calculation of each order in its turn, the results then being employed for the evaluation of terms of the next order (Simon, 1972; henceforth S1).

In the present work we shall choose a particular model of a massive main-sequence star and evaluate its pulsational amplitudes up to and including second-order. The two Fourier coefficients involved here are the constant term and the coefficient of $\cos 2\sigma_0 t$. In S1 these were called B_0 and B_2, respectively, but, here and in what follows, we shall call them u_* and w_* thus bringing the notation into line with that of an earlier work (Simon, 1971).

To derive the second-order pulsation equations, we begin with the stellar equations

$$\frac{d^2 r}{dt^2} + 4\pi r^2 \frac{dP}{dm} + \frac{Gm}{r^2} = 0\,,$$

$$\frac{dm}{dr} = 4\pi r^2 \varrho\,.$$

Inserting

$$r = a(1+\eta)$$
$$\varrho = \varrho_0(1+\varrho_*)$$
$$P = P_0(1+p_*)$$

into the above, we easily obtain equations up to second-order in the small quantities η, ϱ_*, p_*. Here a, ϱ_0 and P_0 represent the equilibrium radius, density and pressure, respectively. To second-order we may seek a solution to the equations mentioned above in the form (S1; Simon, 1971)

$$\eta = x_* \cos\sigma_0 t + w_* \cos 2\sigma_0 t + u_*\,,$$

where x_* is the first-order amplitude, and σ_0 the angular frequency, corresponding to the fundamental mode; w_* and u_* are corrections $\approx x_*^2$. Using the above expression for η, we find (Simon, 1971)

$$\varrho_* = \varrho_{*1} \cos\sigma_0 t + \varrho_{*2} \cos 2\sigma_0 t + \varrho_{*0}\,, \tag{1}$$

where[1])

$$\begin{aligned}
\varrho_{*1} &= -(3x_* + a x_*') \\
\varrho_{*2} &= 3x_*^2 + \tfrac{1}{2} a^2 (x_*')^2 + 2 a x_* x_*' - 3 w_* - a w_*' \\
\varrho_{*0} &= 3x_*^2 + \tfrac{1}{2} a^2 (x_*')^2 + 2 a x_* x_*' - 3 u_* - a u_*'
\end{aligned} \tag{2}$$

(primes indicate differentiation with respect to a) and

$$p_* = \Gamma_1 \varrho_* + \Theta_1 \varrho_*^2\,,$$
$$\Theta_1 = \tfrac{1}{2} \Gamma_1 [\Gamma_1 - 1 + (\partial \ln \Gamma_1 / \partial \ln \varrho)_S]\,.$$

[1]) Note that ϱ_{*1} was called ϱ_* in Simon (1971).

Finally, equating coefficients of Fourier terms, we obtain uncoupled differential equations for w_* and u_* (S1). These equations look formally just like (7.11) and (7.12) of Rosseland (1949), and become identically the latter (with $\Gamma_1 \to \gamma$) under the following approximations

$$\Gamma_1' = \Theta_1' = 0; \quad \Theta_1 = \tfrac{1}{2}\Gamma_1(\Gamma_1 - 1).$$

(Justification for these relations shall be offered in the Appendix). Thus, we have finally (Rosseland, 1949; Simon, 1971)

$$4\varrho_0\, a\sigma_0^2 w_* + L(w_*) = \varrho_0\, a\sigma_0^2 x_*^2 + \tfrac{1}{2} Q(x_*), \tag{3}$$

$$L(u_*) = \varrho_0\, a\sigma_0^2 x_*^2 + \tfrac{1}{2} Q(x_*). \tag{4}$$

II. The Model and the Linear Terms

We shall now solve Eqs. (3) and (4) for a specific stellar model, which we take to be the 121.1 $M_\odot$ star of Schwarzschild and Härm (1958). An equilibrium model for this star was kindly furnished to us by Dr. R. Stothers. Table 1 gives values for some of its important parameters.

Table 1. Characteristics of the unperturbed model

$M/M_\odot$	121.1
β_c	0.502
$\log T_c$	7.656
$\varrho_c/\bar{\varrho}$	22.94
$\log(L/L_\odot)$	6.254
$\log(R/R_\odot)$	1.121
$\beta_{\rm sur}$	0.627
$\sigma_0^2 R^3/GM$	2.208

The linear pulsation equations for this model were solved in the usual way, yielding first-order amplitudes x_* and ϱ_{*1} approximately equal to those obtained by Schwarzschild and Härm (1959). The square of the dimensionless angular frequency of the fundamental mode is given in the last line of Table 1. The luminosity amplitude, on the other hand, was calculated following Boury *et al.* (1964) according to the scheme in which core convection does not adapt to the pulsation. Thus, with

$$L_r = L_{0r}(1 + l_*)$$

$$l_* = l_{*1}\cos\sigma_0 t + l_{*2}\cos 2\sigma_0 t + l_{*0},$$

we find

$$l_{*1} = l_{*1}(\mathrm{rad}) f_{\rm rad} + l_{*1}(\mathrm{turb})(1 - f_{\rm rad}),$$

where

$$l_{*1}(\mathrm{rad}) = 4[x_* + (\Gamma_3 - 1)\varrho_{*1}] + \frac{(\Gamma_3 - 1)}{d\ln T_0/da}\frac{d\varrho_{*1}}{da}$$

$$l_{*1}(\mathrm{turb}) = 2x_* + \Gamma_3\varrho_{*1}$$

and

$$f_{\rm rad} = \begin{cases} L_r(\mathrm{rad})/L_r \text{ in the convective core} \\ 1 \text{ elsewhere}. \end{cases}$$

III. The Second-Order Terms

Equation (3) was solved by Eddington (1919) and Kluyver (1935) for a specific polytropic model, and, more recently, by Simon and Sastri (1972; henceforth SS) for a variety of polytropes. Equation (4) was also solved for a variety of polytropes by Simon (1971). Unfortunately, an error in the computer program in the latter work rendered many of the numerical results invalid. These results are corrected elsewhere (SS).

Figure 1 shows the plots of w_* and u_* versus normalized radius x for the present model. The comparative behavior of these functions – in particular, the much sharper fall-off of w_* from the surface inward – is discussed in detail in SS.

The second-order density variations have already been given explicitly in terms of the radius amplitudes. With

$$T = T_0(1 + t_*),$$

we easily find to second order for adiabatic pulsations

$$t_* = (\Gamma_3 - 1)\varrho_* + \Theta_3\varrho_*^2$$

where

$$\begin{aligned}\Theta_3 &\equiv \tfrac{1}{2}(\Gamma_3 - 1)[\Gamma_3 - 2 + (\partial\ln(\Gamma_3 - 1)/\partial\ln\varrho)_S] \\ &\simeq \tfrac{1}{2}(\Gamma_3 - 1)(\Gamma_3 - 2)\end{aligned} \tag{5}$$

(see the Appendix).

Then, setting,

$$t_* = t_{*1}\cos\sigma_0 t + t_{*2}\cos 2\sigma_0 t + t_{*0},$$

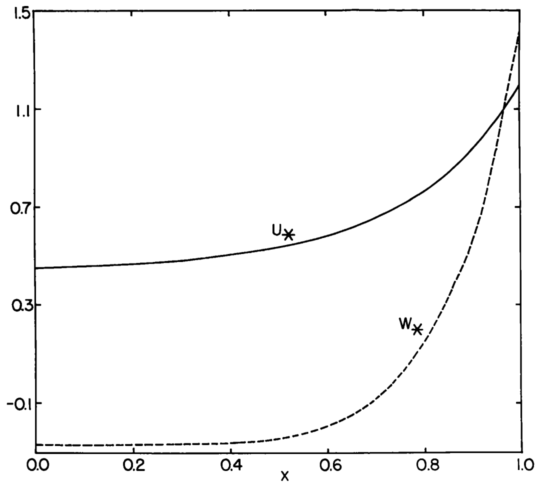

Fig. 1. Second-order radial amplitudes versus normalized radius. Solid curve: u_*; dashed curve: w_*

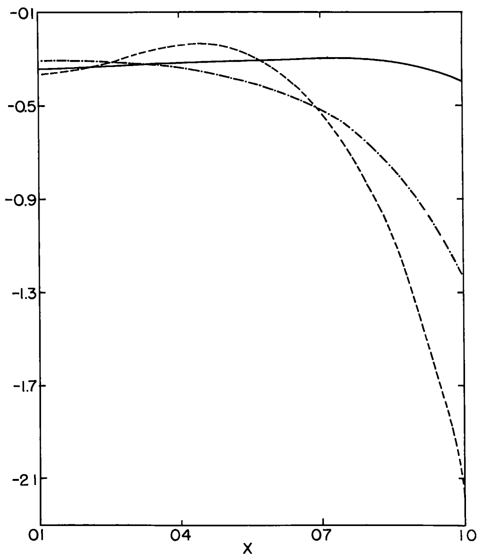

Fig. 2. Second-order time-independent corrections to physical variables. Solid curve: ϱ_{*0}; dashed curve: l_{*0}; dash-dot: t_{*0}

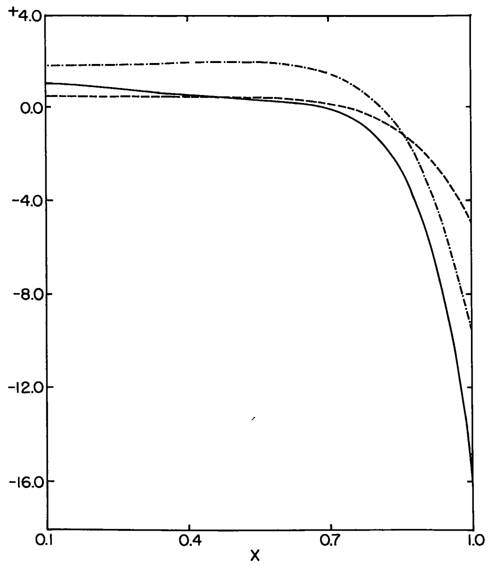

Fig. 3. Second-order anharmonic corrections to physical variables. Solid curve: l_{*2}; dashed curve: t_{*2}; dash-dot: ϱ_{*2}

and using Eq. (1):

$$t_{*1} = (\Gamma_3 - 1)\,\varrho_{*1}$$
$$t_{*2} = (\Gamma_3 - 1)\,\varrho_{*2} + \tfrac{1}{2}\,\Theta_3\,\varrho_{*1}^2$$
$$t_{*0} = (\Gamma_3 - 1)\,\varrho_{*0} + \tfrac{1}{2}\,\Theta_3\,\varrho_{*1}^2\,,$$

where ϱ_{*1}, ϱ_{*2} and ϱ_{*0} are given in Eq. (2).

We now turn to the second-order variations in luminosity. In the envelope, these may be derived directly from the equation of radiative transfer. In the core, convection must be taken in account, and this has again been done according to the non-adaptive scheme of Boury *et al.* (1964). In the approximation that all variations and space derivatives of β may be ignored, as well as the differences between the value of any of the Γ's inside a turbulent element and its value in the surroundings, we obtain

$$l_{*k} = l_{*k}\,(\mathrm{rad})\,f_{\mathrm{rad}} + l_{*k}\,(\mathrm{turb})\,(1 - f_{\mathrm{rad}})$$
$$k = 0,2$$

where

$$\begin{aligned} l_{*k}\,(\mathrm{rad}) = {} & 4v_* + 3x_*^2 + [2\Theta_3 + 3(\Gamma_3 - 1)^2]\,\varrho_{*1}^2 \\ & + 4(\Gamma_3 - 1)\,\varrho_{*k} \\ & + 8(\Gamma_3 - 1)\,x_*\varrho_{*1} + \frac{1}{d\ln T_0/da}\{(\Gamma_3 - 1)\,\varrho'_{*k} \\ & + [\Theta_3\varrho_{*1} + 1.5(\Gamma_3 - 1)^2\,\varrho_{*1} \qquad (6) \\ & + 2(\Gamma_3 - 1)\,x_*]\,\varrho'_{*1}\}\,, \\ l_{*k}\,(\mathrm{turb}) = {} & 2v_* + \tfrac{1}{2}\,x_*^2 + \tfrac{1}{2}\,[\Theta_3 + (\Gamma_3 - 1)]\,\varrho_{*1}^2 \\ & + \Gamma_3\,[\varrho_{*k} + x_*\varrho_{*1}]\,, \end{aligned}$$

where

$$\varrho'_{*k} = 8x_*x'_* + 2ax_*x''_* + a^2x'_*x''_* + 3a(x'_*)^2 - 4v'_* - av''_*\,,$$

$$v_* \equiv \begin{cases} w_* & \text{for } k = 2 \\ u_* & \text{for } k = 0 \end{cases}$$

and Θ_3 is given by Eq. (5).

Figure 2 gives a plot through the star of the time-independent components ϱ_{*0}, t_{*0} and l_{*0}, while Fig. 3 shows ϱ_{*2}, t_{*2} and l_{*2}, i.e., the terms varying like $\cos 2\sigma_0 t$. The central and surface values of the above functions, as well as some others of interest, are provided in Table 2. Note that all quantities are normalized to a surface value of x_* equal to unity. The linear amplitudes scale like $x_{*|\mathrm{sur}}$, the second-order amplitudes like $x_{*|\mathrm{sur}}^2$ (Simon, 1971).

Noting that the ordinate scales of Figs. 2 and 3 are quite different, one is immediately struck by the homologous nature of the time independent terms (which

Table 2. Characteristics of the pulsating model

	Center	Surface
x_*	0.583	1.00
ϱ_{*1}	-1.75	-4.26
t_{*1}	-0.648	-1.67
l_{*1}	-1.16	-2.67
u_*	0.454	1.19
ϱ_{*0}	-0.343	-0.400
t_{*0}	-0.305	-1.24
l_{*0}	-0.380	-2.18
w_*	-0.268	1.45
ϱ_{*2}	1.82	-9.94
t_{*2}	0.497	-4.97
l_{*2}	1.06	-16.1

are functions of u_*) compared with the $\cos 2\sigma_0 t$ terms (functions of w_*). The seat of this behavior turns out to be the previously mentioned sharp fall-off of w_*, as compared with u_*, from the surface inward. Thus while the u_* (i.e., subscript "0") terms are quite moderate in size, we note from Table 2 that a very modest surface value of w_* (a quantity which, we reiterate, scales like $x^2_{*|\mathrm{sur}}$) produces large corresponding variations in density (ϱ_{*2}) and luminosity (l_{*2}). What this may mean is that, so far as the extended stability integral is concerned, the pulsation becomes nonlinear for relatively small values of radius amplitude x_*. If it should also happen that the overall tendency of the w_* terms (subscript "2") is to produce a damping effect, we might expect pulsations to be stabilized at a relatively small amplitude.

In connection with the above it becomes interesting to note the following. First, for zero-age main-sequence stars, a resonance occurs between the fundamental and the next highest mode, somewhere around 55 $M_\odot$ depending on composition (Murphy, 1968). At this point the sign of $w_{*|\mathrm{sur}}$ changes from negative to positive and remains that way as one moves up the main-sequence until the supermassive regime is reached. Thus from $\simeq 55\,M_\odot$ up until at least a few hundred $M_\odot$, w_* and w'_* are positive at the surface (SS), and ϱ_{*2} and l_{*2} exhibit qualitatively the same behavior as in the present case. But since this is just the mass range where pulsational instabilities in main-sequence stars are expected (Schwarzschild and Härm, 1959; Stothers and Simon, 1970) but not observed, it is possible that the w_* terms do indeed exert a damping effect, and that this occurs at amplitudes small enough such that an existing pulsation could go undetected. Whether or not this is actually the case (other explanations for the absence of massive main-sequence pulsations have, of course, been offered, e.g., Larson and Starrfield, 1971) can be determined only by evaluation of the extended stability integral itself.

IV. Resonances in the Model

Table 3 gives dimensionless angular frequencies $\sigma_j(R^3/GM)^{1/2}$ for the first 11 normal modes, along with a tabulation of integer multiples of the fundamental. We can expect a resonance effect whenever any given entry in the last column of Table 3 is approximately equal to an entry in the second column (for $j=1$–3) or the third column (for $j=4$–10). Two such cases may be identified: $j=3$, $k=4$; and $j=8$, $k=8$. On the basis of arguments advanced in S1 we shall ignore the latter without discussion. On the other hand, regarding the first of these, the work in SS suggests that here the resonance might be strong enough to have a non-negligible effect. This effect would occur in the coefficient of $\cos 4\sigma_0 t$, a term which is ordinarily of order x^4_* and thus for our purposes negligible (S1). Whether or not the resonance is actually strong enough to force us to take account of this term depends upon the scale we choose for x_*, i.e., upon the amplitude of the pulsation. Furthermore, even given a specific amplitude, an accurate estimate of the resonant properties would require dealing with an equation of fourth-order in x_* – an exercise well beyond the scope of the present investigation. In view of these uncertainties, let us agree that here, and in all subsequent articles of this series, we shall ignore entirely the existence of this resonance, realizing that our calculations will be inaccurate in proportion to the strength of the resonant effect. Thus the results we obtain, while perhaps not fully treating the properties of our present specific model, will nonetheless accurately describe the nonresonant behavior of a massive main-sequence star; and, since low-order resonances are far from inevitable in such stars (note e.g., the $\mu^2 M/M_\odot = 30$ model in Murphy, 1968), we may be confident that we are still tackling the major part of our problem.

Table 3. The normal mode frequencies

j	$\sigma_j(R^3/GM)^{1/2}$ Calculated	Predicted[a])	k	$k\sigma_0(R^3/GM)^{1/2}$
0	1.486	–	–	–
1	3.400	2.229	2	2.972
2	4.701	4.533	3	4.458
3	5.951	5.876	4	5.944
4	–	7.141	5	7.430
5	–	8.331	6	8.916
6	–	9.521	7	10.40
7	–	10.71	8	11.89
8	–	11.90	9	13.37
9	–	13.09	10	14.86
10	–	14.28	11	16.35

[a]) According to formulas given in SS. Note that the predicted value is already very good for $j=3$.

V. Prospectus

In future papers of this series, we shall carry the calculation to third order in the adiabatic amplitudes, propose a scheme for taking into account the effects of pulsationally-induced changes in the stellar entropy distribution, and finally write down and evaluate the extended pulsational stability integrals themselves. With the completion of this program, comparisons with existing hydrodynamic pulsation calculations for massive stars (e.g., Appenzeller, 1970; Talbot, 1971; Ziebarth, 1971) will hopefully be possible, including some attempt at evaluating the various artificial boosting schemes employed in the hydrodynamic work.

Acknowledgements. The author gratefully acknowledges partial support by a Summer Research Fellowship from the Research Council, University of Nebraska, Lincoln.

Appendix

Approximations Regarding Radiation Pressure

The approximations

$$\Theta_1 \simeq \tfrac{1}{2}\Gamma_1(\Gamma_1-1)$$
$$\Theta_3 \simeq \tfrac{1}{2}(\Gamma_3-1)(\Gamma_3-2)$$

are easily justified by a calculation of the errors involved. For the former, the error incurred over the whole range of possible values of β is always $<4\%$ (Simon, 1970), while, for the range of the present model, $0.502 \leqq \beta \leqq 0.627$, the error is well below 2%. In the case of Θ_3, although the error may exceed 16% for certain β's, over the range of our model it is about 3.5% at most. We shall now justify neglecting the space derivative of β, and thus also the derivatives of Γ_1 and Θ_1. We have

$$\frac{d\beta}{da} = (\beta-1)\frac{d\ln P_0}{da}\left[4\frac{d\ln T_0}{d\ln P_0}-1\right] \equiv -A\,\frac{d\ln P_0}{da}.$$

In the convective core

$$A=(1-\beta)\left[\frac{4(\Gamma_2-1)}{\Gamma_2}-1\right]<0.04$$

for all β. Furthermore, it is easy to show that, for the model in question, A is always smaller outside the core than inside it, and thus

$$\left|\frac{d\beta}{da}\right|<0.04\left|\frac{d\ln P_0}{da}\right|$$

everywhere. Hence the neglect of derivatives of β.

Let us now examine the variations of β during the pulsation. With

$$\beta=\beta_0(1+\beta_*),$$
$$\beta_*=\beta_{*1}\cos\sigma_0 t+\beta_{*2}\cos 2\sigma_0 t+\beta_{*0},$$

we find

$$\beta_{*1}=G_1\varrho_{*1}$$
$$\beta_{*k}=G_1\varrho_{*k}+G_2\varrho_{*1}^2$$
$$k=0,2$$

where

$$G_1=\frac{1-\beta}{\beta}[\Gamma_1-4(\Gamma_3-1)]$$
$$G_2=\tfrac{1}{2}[\Theta_1-4\Theta_3-\Gamma_1^2+4\Gamma_1(\Gamma_3-1)-6(\Gamma_3-1)^2].$$

Over the range of our model, we find

$$|G_1|<0.07$$
$$|G_2|<0.02.$$

We shall thus agree to neglect variations of β. Furthermore, on the basis of the smallness of the above calculated changes in β, and following Boury *et al.* (1964), we shall also ignore any differences between the value of a Γ inside a turbulent element and its value outside. We close this Appendix by posing the following problem. Although it is true that, within each order, terms arising from changes in β are small compared with other terms, is it not the case that a first-order "β-change term" (e.g., β_{*1}) might be as large as an ordinary second-order term (e.g., ϱ_{*1}^2) thus rendering neglect of the former invalid? The answer to this question is that the whole nature of our approach is that all terms should be segregated according to their order, and size comparisons made only within a given order (S1). It is quite correct that at some given small amplitude a term like β_{*1} might completely dwarf one like ϱ_{*1}^2. However, our interest is entirely in what happens as the pulsation grows, and since, in this process, the higher-order terms are always competing with the *leading* terms of lower order, our neglect of the smaller terms within each order is wholly justified.

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