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Resonance Effects in Polytropes

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Summary. Second-order anharmonic pulsational amplitudes are investigated for a variety of polytropic models. It is found that the qualitative behavior of these amplitudes is decisively affected by the existence of resonances – this remaining true even when a given model is itself not in a resonant situation. An empirical asymptotic formula is given for the normal mode spectra of polytropes. The question of resonances is discussed for real stars, and a suggestion made for classifying Cepheid-type pulsators on the basis of resonant properties. Finally, invalid numerical results from an earlier work (Simon, 1971) are corrected in the Appendix.

Key words: nonlinear pulsations – resonances – anharmonic pulsations

I. Introduction

In an approach to nonlinear pulsations due to Eddington (1919), the relative radial amplitude is expanded in the form

$$\eta = x_* \cos \sigma t + w_* \cos 2\sigma t + u_*,$$

where x_* is the linear term, and w_* and u_* are corrections of second-order which satisfy uncoupled differential equations. The time-independent component u_* was investigated in an earlier paper (Simon, 1971). However, an error in the computer program connected with this work resulted in the publication of incorrect numerical results. This error has now been rectified and some corrected results appear in the Appendix.

In the main body of the present work we shall be interested in the quantity w_*, which satisfies the second-order (i.e. $\approx x_*^2$) equation

$$4\varrho_0 a \sigma_0^2 w_* + L(w_*) = \varrho_0 a \sigma_0^2 x_*^2 + \tfrac{1}{2} Q(x_*), \qquad (1)$$

where L is an operator appearing in the linear equation

$$L(\xi) + \varrho_0 a \sigma^2 \xi = 0, \qquad (2)$$

and Q contains quantities of second-order; both L and Q are given explicitly in Rosseland (1949). As is well known, Eq. (2) yields a spectrum of normal modes (ξ_i, σ_i) the ξ_i being orthogonal over the star with respect to $\varrho_0 a^4$ (Ledoux and Walraven, 1958). In Eq. (1) we have used x_* to denote to within a scale factor the eigenfunction of the fundamental mode, ξ_0.

Equation (1) has been integrated for a variety of polytropic models. The procedure involved is similar to that required for u_*, and is described in Simon (1971). (See also Eddington, 1919; Kluyver, 1935). Figure 1 shows a plot of the surface value of w_* vs. the square of the

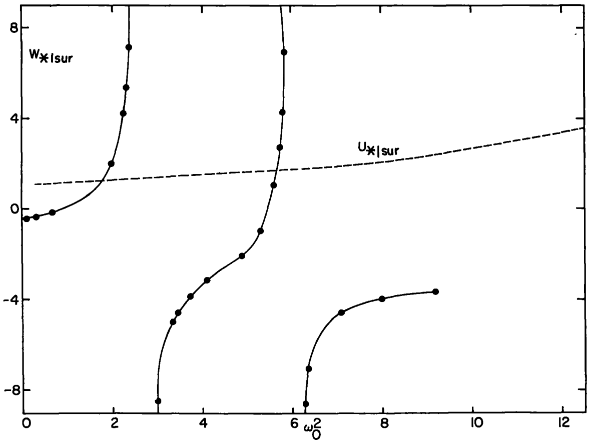

Fig. 1. Surface anharmonic amplitude $w_*|_{sur}$ (points connected by solid lines) versus ω_0^2 for various cases of the standard model. Dashed line represents $u_*|_{sur}$ for the same models

dimensionless angular frequency of the fundamental mode $\omega_0^2 \equiv \sigma_0^2 R^3/GM$, for a number of examples of the standard model (points joined by solid curves), each corresponding to a different value of β (ratio of gas pressure to total pressure). The dotted line represents schematically the run of surface values of u_* for the same models. One notices at once the erratic behavior of $w_*|_{sur}$ compared with $u_*|_{sur}$. This behavior, surprising at first glance, can be explained quite readily in terms of the existence of two resonances which dominate the behavior of w_*. To see this we need merely expand w_* in terms of the complete set of linear eigenfunctions, i.e.,

$$w_* = \sum_i b_i \xi_i.$$

Inserting this in Eq. (1), multiplying through by $a^3\xi_j$, and integrating, we obtain

$$b_j = \frac{1}{(4\sigma_0^2 - \sigma_j^2)} \int_0^R F(a)\xi_j da \tag{3}$$

where

$$F(a) = \varrho_0 a^4 \sigma_0^2 x_*^2 + \tfrac{1}{2} a^3 Q(x_*)$$

is a known function of a once the linear amplitude x_* has been calculated.

With the solution to Eq. (1) now in familiar form, we see immediately from Eq. (3) that resonant behavior in w_* may be expected whenever any $\sigma_j \simeq 2\sigma_0$. In the standard model this occurs twice: for $\omega_0^2 \simeq 2.64$ ($\beta \simeq 0.40$) with $j=1$; and for $\omega_0^2 \simeq 5.96$ ($\beta \simeq 0.81$) with $j=2$. The former case is the polytropic analogue of the resonance mentioned by Murphy (1968).

II. The Nonresonant Behavior of w_*

The striking fact emerging from Eq. (3) (and obvious from a glance at Fig. 1) is that even quite far from a resonant model the qualitative behavior of w_* is dominated by the existence of the resonances. Table 1 gives

Table 1. Anharmonic terms for selected models[a])

Variable	$n=1.5$		$n=3.0$		$n=4.5$
	$\beta=1.0$	$\beta=0.9$	$\beta=0.7$	$\beta=0.3$	$\beta=1.0$
ω_0^2	2.71	7.08	4.86	1.98	17.2
$w_*\|_{sur}$	1.48	− 4.69	− 2.07	1.94	− 5.05
$w_*\|_{cen}$	− 0.78	− 0.01	0.01	− 0.22	0.00
$\varrho_{*2}\|_{sur}$	−11.2	97.6	35.3	−13.4	207

[a]) Pulsational quantities normalized to $x_*|_{sur} = 1.0$.

some of the relevant quantities calculated for the standard model ($n=3$) as well as for a number of other polytropes. The symbol ϱ_{*2} denotes the anharmonic correction to the density fluctuation (time dependence, $\cos 2\sigma_0 t$) introduced by the second-order terms. It is given explicitly in Simon (1971). The runs of w_* for two typical nonresonant models, and the run of u_* for one model, are plotted in Fig. 2.

As one can see from the example in Fig. 2, the surface value of u_* turns out to be positive as is its first derivative. Furthermore this is the case for all models calculated (see the Appendix). To obtain additional information we make the further expansion

$$u_* = \sum_i c_i \xi_i .$$

Inserted in the differential equation for u_* (Appendix) this yields

$$c_j = -\frac{1}{\sigma_j^2} \int_0^R F(a)\xi_j da. \tag{4}$$

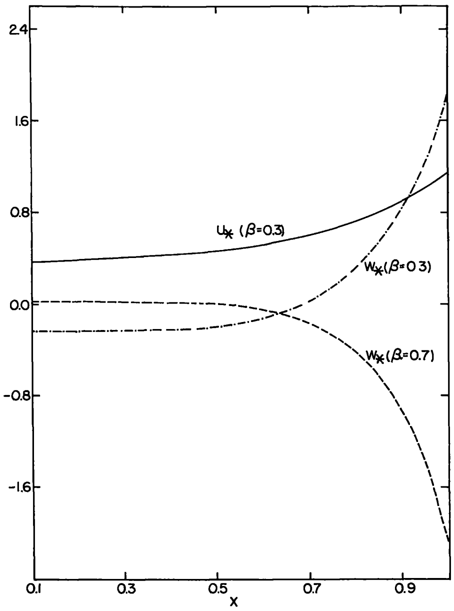

Fig. 2. Second-order amplitudes u_*, w_* versus normalized radius x for two cases of the standard model. Solid line: u_* ($\beta=0.3$); dashed line: $w_*(\beta=0.7)$; dash-dot: $w_*(\beta=0.3)$

Now, one can easily verify from Rosseland (1949) that $F(a)$ is a function which vanishes at the surface and center of the model, peaking somewhere in the stellar envelope. On the other hand, it is well known that the ξ_j are positive when large, and that they fall off faster and faster from the surface inward as j gets larger. The latter behavior, along with the presence of the factor σ_j^2 in the denominator of Eq. (4), insures that c_j will be large only for small j. In fact, it is the fundamental linear mode which makes the major contribution to u_*, a result which tells us at once that $F(a)$ is preponderantly negative over the star.

The behavior of w_* is, of course, more complicated. We first note the far faster drop-off from the surface inward of $|w_*|$ as compared with u_*. The reason for this can be found in the expression $(4\sigma_0^2 - \sigma_j^2)^{-1}$ appearing in Eq. (3), as opposed to σ_j^{-2} in Eq. (4). The latter decreases monotonically with increasing j, while the former does not. What this means is that modes higher than the fundamental can contribute much more strongly to w_* than they can to u_* – hence the much steeper slope (characteristic of the higher modes) of the former. In particular, that ξ_j corresponding to the σ_j lying closest to a resonance will make the largest contribution (provided j is not large, since in that case the integral in Eq. (3) will be small), and the sign of that contribution (i.e., the sign of b_j) will depend on whether $(4\sigma_0^2 - \sigma_j^2)^{-1}$ is positive or negative.

In this manner the dual nature of the behavior of w_* – namely w_*, dw_*/da positive at the surface or w_*, dw_*/da negative at the surface – is easily explained: it merely depends on which side of the nearest resonance the model happens to be sitting! Furthermore, since ϱ_{*2} turns

out to be dominated by a term proportional to dw_*/da (such that a positive derivative gives a negative ϱ_{*2} at the surface, and vice versa) we have the situation that not only the second-order asymmetry of the velocity curve, but the anharmonic density correction as well, has its sign determined by the position of the nearest resonance. We shall have more to say about this shortly.

III. The Resonances

We begin this section by defining the "distance" of a given model from the resonant situation $2\omega_0 = \omega_j$; thus

$$Z_j \equiv 4\omega_0^2 - \omega_j^2 .$$

Plotted in Fig. 3 is $w_*|_{\text{sur}}$ vs. Z_1 for models falling in the vicinity of the resonance near $\omega_0^2 \simeq 2.64$, while Fig. 4 gives $w_*|_{\text{sur}}$ vs. Z_2 in the region near $\omega_0^2 \simeq 5.96$. Vertical lines in these figures represent "halfwidths" of the resonances, determined as follows.

Compare for a moment Eqs. (3) and (4); they differ only in the coefficient multiplying the integral. Since it is just this coefficient, which, in the case of w_* [Eq. (3)] carries the resonant behavior, but in the case of u_* [Eq. (4)] does not, it is reasonable to say that, in a given model, u_* represents in some sense the size of the nonresonant behavior of the second-order terms. Defining things thus, we shall now agree, somewhat arbitrarily, that a model will be said to fall "inside" a resonance whenever $|w_*| \geqq 10u_*$. The halfwidths are then given by models on the boundaries of the resonance, i.e., for $|w_*| = 10u_*$, $w_* > 0$ and $|w_*| = 10u_*$, $w_* < 0$.

With the above definition we find halfwidths of $|Z_1| \simeq 0.55$ for $j=1$, and $|Z_2| \simeq 0.25$ for $j=2$. The fact that the halfwidth is considerably larger for $j=1$ than it is for $j=2$ is readily explained in terms of the integral in Eq. (3) being larger for smaller j. In general, the lower the normal mode picked-up resonantly by the fundamental, the wider will be the resonance.

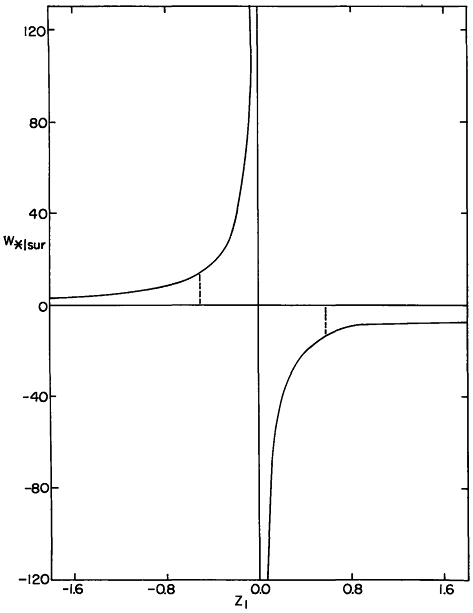

Fig. 3. A close-up view of the resonance near $\omega_0^2 = 2.64$. Vertical lines mark off the "halfwidths"

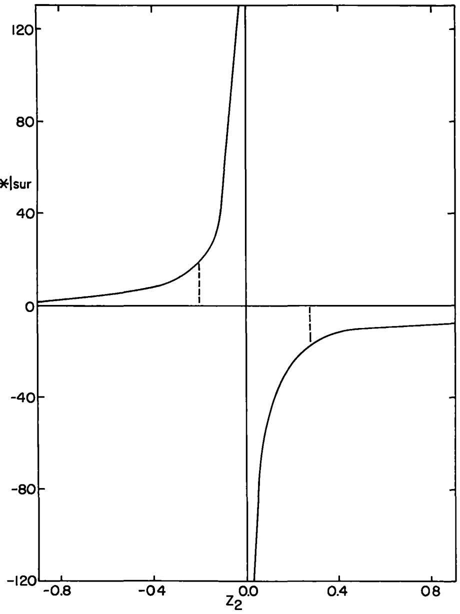

Fig. 4. A close-up view of the resonance near $\omega_0^2 = 5.96$. Vertical lines mark off the "halfwidths"

IV. An Empirical Formula

Up to now, we have discussed a rather special case of a resonance, namely that where one of the normal modes has a frequency approximately twice the fundamental. A much more general class of resonant effects may be defined by means of the quantity

$$Z_{kj} = k^2\omega_0^2 - \omega_j^2 .$$

Here, of course, we expect a resonance between the fundamental and a higher mode whenever $Z_{kj} \simeq 0$ for any integer pair (k,j), where $k>1, j \geqq 1$.

Since it has already been shown in the case treated that the existence of resonances will have qualitative effects on the pulsation even where the resonant effect itself is not very strong, it becomes of interest to find some means of discovering the presence of resonances in given specific models. To do this, of course, requires knowledge of the spectrum of normal modes.

Asymptotic formulae for the normal mode frequencies have already been derived by Ledoux (1962) and Van der Borght (1964). In this section we shall present a similar (but empirical) relation which is somewhat more accurate than the above, particularly for the lower modes. The formula is as follows:

$$\frac{\omega_j}{\omega_{j-1}} = \frac{j+b}{j+b-1}, \tag{5}$$

Table 2. Calculated and predicted values for the normal modes

	$n=1.5,\ \beta=1.0$						$n=2.0,\ \beta=1.0$					
j	0	1	2	3	4	5	0	1	2	3	4	5
ω_j (calc.)	1.645	3.540	5.156	6.702	8.208	9.703	2.000	3.653	5.156	6.613	8.044	9.459
ω_j (pred.)	–	3.290	5.310	6.875	8.378	9.850	–	3.587	5.269	6.738	8.165	9.573
% error	–	7.1	3.0	2.6	2.1	1.5	–	1.8	2.2	1.9	1.5	1.2
	$n=3.0,\ \beta=0.40$						$n=4.5,\ \beta=1.0$					
j	0	1	2	3	4	5	0	1	2	3	4	5
ω_j (calc.)	1.626	3.251	4.524	5.753	6.960	8.155	4.147	5.219	6.295	7.374	8.455	9.534
ω_j (pred.)	–	2.439	4.335	5.655	6.904	8.120	–	5.184	6.263	7.344	8.427	9.512
% error	–	25	4.2	1.7	0.80	0.43	–	0.67	0.51	0.41	0.33	0.23

where

$$b=\tfrac{1}{2}2^{2n/3} \tag{6}$$

and n is the polytropic index. Table 2 gives calculated and predicted frequencies for the first few modes in four specific cases: $n=1.5$ ($\beta=1.0$), $n=2$ ($\beta=1.0$), $n=3$ ($\beta=0.40$), $n=4.5$ ($\beta=1.0$). In each case any given predicted frequency ω_j was gotten from the preceding *calculated* [via integration of the linear Eq. (2)] frequency ω_{j-1} using Eqs. (5) and (6).

The rapid convergence of the formula is evident from Table 2. Furthermore this formula turns out to be valid (with $n=3$) for real stars on the main-sequence and in early post-main-sequence evolution (see also, Van der Borght, 1964). This is of course due to the fact that the envelopes of such stars look like the standard model (Stothers, 1965). Since the envelopes of other real stars, e.g., Cepheids, also look like polytropes (Christy, 1966), it is not unreasonable to expect that formulae *like* Eqs. (5) and (6) will have a wide validity.

Once Eq. (5) has nearly converged for a given model, we may with small error calculate the jth frequency from the known ith frequency as follows:

$$\omega_j=\omega_i\,\frac{j+b}{b+i}\,.$$

Rewriting this as

$$\frac{\omega_j}{\omega_0}=a_1 j+a_2\,,$$

where

$$a_1=\frac{\omega_i}{\omega_0(b+i)} \quad \text{and} \quad a_2=\frac{\omega_i b}{\omega_0(b+i)}$$

both turn out to be positive, irrational (since ω_i/ω_0 is, in general, irrational) numbers of order unity, we see that a resonance appears whenever the equation

$$k=a_1 j+a_2 \tag{7}$$

is satisfied for an integer pair (k, j). Since k and j both form infinite sets, we expect that Eq. (7) may be satisfied resonantly to any accuracy required. The implications of this fact for a theory of nonlinear pulsations are discussed elsewhere (Simon, 1972).

V. Discussion

As a star evolves, its spectrum of normal modes changes continuously, so that an evolving stellar model will always be moving in and out of resonant regions. To get some estimate of how long a given star might remain inside a resonance, we shall examine some particular cases – namely the 15, 20 and 30 $M_\odot$ models of Stothers (1963, 1965). In the course of their main-sequence evolution these stars traverse the resonance $k=2$, $j=2$. Figure 5 shows an H–R diagram for the above stars,

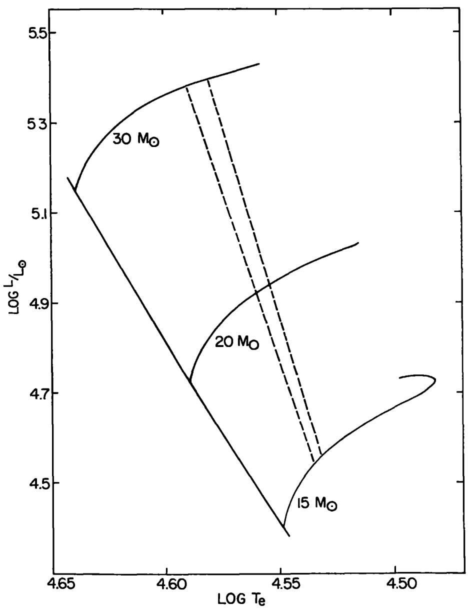

Fig. 5. Evolution of massive main-sequence stars (Stothers, 1963; 1965) through the resonance band $k=2$, $j=2$ (dotted lines). Main-sequence is drawn in at left

with their evolutionary tracks (solid lines) crossing the $k=2, j=2$ resonance band (dotted lines). On the basis of evolution times published by Stothers, and adopting the resonance halfwidth $|Z_2| \simeq 0.25$ determined above, we find that these stars spend roughly 4–5% of their main-sequence lifetimes inside the resonance. Since $|Z_1| \simeq 2|Z_2|$, we may then estimate that a main-sequence star evolving through the resonance $k=2, j=1$ would spend about 10% of its lifetime within the resonance band.

Now, of course, a medium-mass main-sequence star is not pulsationally unstable, so that the question of whether it is or is not inside a resonance at a given time is quite academic. On the other hand, as estimated above, the fraction of stellar lifetime spent in a resonant region is just large enough such that the possibility arises of perhaps catching an observed pulsator within such a resonance band. In particular, it would be interesting to calculate resonance lifetime fractions for stars crossing the Cepheid instability strip at various luminosities to see whether or not such observations are statistically feasible for these stars. (Since in the case of Cepheid-type pulsators, it is well known that higher normal modes, in particular the first "overtone", are capable of independent energization, the analysis would need to be expanded to include resonances which need not involve the fundamental).

Furthermore, we note here that it might not even be necessary to actually observe a pulsator inside a resonance in order for this sort of analysis to yield some useful information. It has already been shown in Section II that the signs of the anharmonic ($\cos 2\sigma_0 t$) second-order corrections w_* and ϱ_{*2} are determined merely by the position of a given model *vis-a-vis* the nearest resonance. Although we do not venture to say on the basis of the present work what effect the signs of these corrections might have on the light or radial velocity curves of specific pulsators (particularly nonadiabatic pulsators), it nonetheless seems reasonable to suggest that observable effects may well be present and are certainly worth seeking.

As a first step in this connection it would be quite easy for groups possessing evolutionary model codes and linear nonadiabatic pulsation codes to demarcate on the H–R diagram the loci of resonances falling within the instability strip. Subsequently, all models could be classified according to where they lie in the H–R diagram with respect to the plotted loci. Comparison of the models with actual pulsators would then indicate whether or not the observed light and velocity curves are distinguishable on the basis of the above classification.

Finally, it is amusing to point out that some pulsating stars will surely, in the course of their evolution, pass directly through the center of a resonance, i.e., through a mathematical discontinuity. Catastrophe, in these cases, can probably be argued away by invoking sobering physical realities. On the other hand, it remains true that a star, having passed through a resonance, should show interesting physical changes occasioned by the expected sign alterations in w_* and ϱ_{*2}. In particular the role played by these quantities in the pulsational stability of the star will, in some cases, reverse as the center of the resonance is crossed, such that what were damping terms will become energizing and vice versa. The questions of how the star could physically accommodate such changes, and of what might be their observable results are, however, well beyond the scope of the present work.

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Appendix

Correction of Results Given in Simon (1971)

The second-order time-independent correction u_* is the solution of the equation (Rosseland, 1949)

$$L(u_*) = \varrho_0 a \sigma_0^2 x_*^2 + \tfrac{1}{2} Q(x_*) .$$

A computer error in the previous calculations of u_* had the effect of crucially altering the *slope* of the function while leaving its shape qualitatively the same. Unfortunately, u'_* enters in a decisive manner in the expression for the time-averaged second-order density perturbation $\langle \delta\varrho/\varrho \rangle$, with the result that this quantity as given in Simon (1971) *does not have the correct sign.* Furthermore, since it is the sign of $\langle \delta\varrho/\varrho \rangle$ which determines the sign of the stability integral K_2, the latter, as given in Simon (1971), also has the incorrect sign.

Table A1. Terms involving u_* for selected models[a])

Variable	$n=1.5$	$n=3.0$			
	$\beta=1.0$	$\beta=0.999$	$\beta=0.900$	$\beta=0.300$	$\beta=0.100$
$u_*\vert_{sur}$	1.42	2.44	2.04	1.16	1.14
$u_*\vert_{cen}$	0.68	0.01	0.02	0.38	0.84
$\langle\delta\varrho\vert\varrho\rangle_{sur}$	−1.49	− 7.81	− 4.56	−0.11	−0.23
K_2	1.17	0.04	0.05	0.22	0.37
L_P/L	−8.64	−51.5	−22.0	−1.32	1.11

[a]) Normalized to $x_*|_{sur}=1.0$.

Table A1 gives for selected models the correct values for $u_*|_{sur}$, $u_*|_{cen}$, $\langle \delta\varrho/\varrho \rangle_{sur}$, K_2 and the total stability integral L_P/L. The effect of the sign changes mentioned above is to render invalid nearly every qualitative statement made in Simon (1971). We now correct these statements briefly as follows.

(1) The preponderant sign of $\langle \delta\varrho/\varrho \rangle$ over the integral K_2 is always negative. (2) K_2 is always an energizing term for luminosity increasing outward, and a damping term for luminosity decreasing outward. (3) The thermal-imbalance integrals I_2, J_2 and K_2 always complement each other, i.e., they have the same sign. (4) In the case of overwhelming radiation pressure (e.g., $\beta=0.10$) a star may become pulsationally unstable due to thermal

imbalance. (5) The results of Okamoto (1967) regarding pre-main-sequence pulsational instability are vindicated insofar as the thermal-imbalance terms are concerned. Finally, we note that errors discussed above occurred in the machine integrations only; all equations in Simon (1971) are correct as written.

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