8-2015

The Strict Higher Grothendieck Integral

Scott W. Dyer
University of Nebraska-Lincoln, koyemsi@yahoo.com

THE STRICT HIGHER GROTHENDIECK INTEGRAL

by

S. W. Dyer

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Mark Brittenham and Susan Hermiller

Lincoln, Nebraska

August, 2015

THE STRICT HIGHER GROTHENDIECK INTEGRAL

S. W. Dyer, Ph.D.

University of Nebraska, 2015

Adviser: Mark Brittenham and Susan Hermiller

This thesis generalizes A. Grothendieck's construction, denoted by an integral, of a fibered category from a contravariant pseudofunctor, to a construction for n- and even ∞-categories. Only strict higher categories are considered, the more difficult theory of weak higher categories being neglected. Using his axioms for a fibered category, Grothendieck produces a contravariant pseudofunctor from which the original fibered category can be reconstituted by integration. In applications, the integral is often most efficient, constructing the fibered category with its structure laid bare. The situation generalizes the external and internal definitions of the semidirect product in group theory: fibration is the internal notion, while the integral is a form of the external semidirect product.

The strict higher integral functor is continuous, and under mild assumptions the integral n-categories produced are complete. The integral retains most formulae (like Fubini's theorem) familiar from analytic geometry, providing a useful calculus for many applications in pure mathematics.

DEDICATION

In memoriam Steve Haataja, David Klarner and Alan Turing, and with special gratitude to all the forgotten souls who have made this work possible.

Contents

Chapter 1

Introduction

All the higher categorical structures in this paper are of the strict variety, and the adjective "strict" will be consistently suppressed. It is important to state this clearly and early for the cognoscenti, but chapter 2 provides an introduction to higher categories suitable for pure mathematicians, emphasizing especially those notions on which this paper most relies. Strict higher category theory is an historically important special case, almost always very well understood before generalization to the case of weak higher category theory.

In his study of descent [4], A. Grothendieck originated the notion of a fibred category, which led to much effort to generalize this to higher categories. Most notably, C. Hermida [5] has illuminated the structure of the 2-category of fibred categories using a definition for a fibred 2-category which admits good properties.

In his original study of fibration (to study descent), Grothendieck furnished a construction which produces any fibred category from a contravariant, category-valued, weak functor. This construction, most often denoted by an integral, generalizes the external semidirect product of groups, and serves as an alternate characterization of fibred categories. It is, perhaps, remarkable that this constructive approach seems

to have been mostly neglected by those seeking higher-dimensional generalizations. While often referred to simply as the Grothendieck construction, I shall refer to it as the Grothendieck integral, or simply the integral, despite its purely algebraic definition. (It is not the construction of the Grothendieck group, often also called simply "the Grothendieck construction.")

The central result of this paper is a generalization of the integral to strict higher categories, but all the good behaviors exhibited by the construction are extensions of this result. The basic construction occupies chapter 3, but a summary of the basics is the content of section 3.8, included for easier reference. To mitigate the length and technicality of the construction, I have tried to include informative examples from a variety of pure mathematical persuits as early in the exposition as possible; I only hope they do not distract. No example is necessary to understand the sequel. Sections 3.9 and 3.10 may be omitted without real loss of continuity. Section 3.9 shows that the integral does not demand that we write the inflation and truncation functors usually neglected notationally (as embeddings and forgetful functors generally are). Section 3.10 gives a detailed description of the cell-structure of the resulting higher integral categories, but the contrasting perspective of enriched categories (see chapter 2) seems almost always to be more useful.

In chapter 4, I develop the higher functoriality of the construction, arguably part of the basic result, but also arguably part of its good behavior. It is noteworthy that one may construct higher functors and natural transformations by way of the integral, and not just higher categories. This use of the integral is a key to chapter 3, as well as a hint of the limit properties of chapter 6; avoiding circularity in chapter 3 is one of its main technical complications. On a first reading, a peek at the results of chapters 4 and 6 may make chapter 3 more easily decypherable.

Chapter 5 contains basic structural results more closely related to the historical

origins of integration as an alternate perspective on fibration. In particular, the integrand is an n-category-valued n-functor, which can be understood as a suitably indexed family of n-categories. These n-categories all embed in the integral (5.1.3) and the integral naturally projects onto the index n-category (5.2.1 and 5.2.2). Perhaps surprisingly, the relationship between fibration and integration becomes unclear (or breaks down) in dimensions $n > 2$, as see the remarks following proposition 5.3.2. This may help explain why prior strong results in the area seem to lie mainly in dimensions $n \leq 2$. Chapter 5 does not furnish results used in the sequel, but the priority of connecting to previous mathematics led to this ordering of the chapters.

Chapters 6 and 7 contain theorems of interest especially in (pure) applications. In proposition 6.2.1, we see that under minimal hypotheses, integral n-categories are complete with a constructive limit structure. As a result, the domain n-categories Int_n and cInt_n of the co- and contravariant integral n-functors are complete this way, and in proposition 6.3.1 we see that both these functors are continuous. (The special case of binary and nullary products is the hint mentioned for chapter 3.) Chapter 7 shows that the order of integration may be reversed (as in Fubini's theorem in measure theory), which is surprisingly useful for a higher categorical understanding of higher categories themselves. The comma categories that Mac Lane [9] calls a "secret weapon in the arsenals of...experts" are example double-integral categories.

Finally, chapter 8 outlines a (parallel) program for generalizing this work to weak higher categories. It is my hope that, in addition to helping to clarify less obviously categorical mathematics, the integral construction may aid mathematicians in finding the right definitions for weak higher category theory, and in understanding the relationships between the many definitions currently available (see [8]).

Justifying a good general theory, like the integral, is subtle, being largely an issue of anecdotal examples. I have tried to include many examples from diverse branches

of mathematics, but many other examples were not included. Some were excessively specialized or technical, but most were omitted merely to avoid diluting the exhibition of my core results. These results would not be hard to duplicate, once given the propositions, but given their constructive nature I have lavished details on them, hoping to save my readers time in putting them to good use in their own research. The slogan should be something like "try integrating every functor you can." I have yet to find an n-category-valued n-functor of an n-category that does not clarify results surrounding its origin; a particularly good example of this is example 3.7.2. Particularly, I would be touched to hear of any other examples my readers may find; should my work prove useful to you, I would ask to share in this joy.

1.1 A Few Specifics

To allow the reader more detail to hold onto when first reading this, consider a category-valued functor $U_\Box$ with domain A, writing U_f in place of the more usual $U(f)$. The classical (1-dimensional) integral category $\int_A U$ contains (isomorphically) each category U_a for each object a of A, interconnecting them according to the functors U_f given by U on each arrow f of A. Specifically, the objects of $\int_A U$ are

$$|\textstyle\int_A U| = \{(a,u) | a \in |A|, u \in |U_a|\},$$

while for objects (a,u) and (b,v), the hom-sets are

$$[\textstyle\int_A U][(a,u),(b,v)] = \{(f,p) | a \xrightarrow{f} b \text{ in } A, U_f u \xrightarrow{p} v \text{ in } U_b\}.$$

The identity at an object (a,u) is $1_{(a,u)} = (1_a, 1_u)$ and the composition of

$$(a,u) \xrightarrow{(f,p)} (b,v) \xrightarrow{(g,q)} (c,w)$$

is $(g,q)(f,p) = (gf, q(U_g p))$; the picture in U_c is

$$U_{gf}u \xrightarrow{U_g p} U_g v \xrightarrow{q} w \,.$$

The idea is that to map u in U_a to v in U_b we first push u into U_b via some transition functor U_f, and then pick an arrow p in U_b from $U_f u$ to v. These details appear later as 3.3.1 where their technicalities will be considered; for now, I have included them for those readers for whom (as for me) specifics are part of the intuitive process.

1.2 Historical Origins and Intuitions

This section is more or less philosophical, and not strictly mathematically necessary. Yet these remarks capture intuitions which I feel complete the mathematical picture.

Every good thing is a (higher) functor, but sometimes this can be less than obvious. For example, the endomorphism monoid at an object in a category C is not a functor of C, but rather of the maximum subgroupoid ΥC of C. In this way, the maximum subgroupoid (2-)functor Υ is important for more than just notational reasons. Avoiding reference to elements has real benefits, but such *constructions* on (n-)categories are important to reveal the (higher) functorialities and naturalities in play. In this thesis I present such a construction, motivated at first by my rebellion against the idea that the semidirect product construction was functorial in one of the groups but not the other. Since then, I have found many examples of it, particularly when good things don't seem functorial. Like a good categorist, I wish I had learned the abstraction before the examples, and so I hope to save the reader time.

One reason to focus on the integral construction is its frequent appearance in pure mathematics outside category theory, where one of the primary roles categories play is as a convenient language. Often, integration is useful in constructing ad hoc domain or codomain categories to express the functoriality of seemingly non-functorial

constructions. It is also often arguably easier to fully construct the right functor to integrate than to construct its fibred category directly and then indirectly recover the functor integrated. This is especially useful when one wishes to pay more attention to the (pure) application than to construction of its domain. As an added bonus, one often obtains unexpected structural information as a side-effect of this approach.

In studying descent, Grothendieck initiated not only the study of fibred categories, but also of pseudofunctors, now more commonly called weak 1-functors. In this way Grothendieck can be credited as one of the first to recognize the value of so-called weak higher categorical structures.

I have heard it said [1] of homological algebra that the best way to learn it is to pick up a book on the topic and prove all the theorems for oneself. While this is perhaps true of any branch of mathematics, the sentiment being expressed is that the results are not hard, but the difficulty lies in knowing what to prove: firstly, it must be true, but even harder, it must be useful. Most of this work has a similar feel, but since the heart of the matter is constructive, I lavish details on the construction. I also detail proofs those results that I found either tricky to establish or especially important. But to really understand something, one has to teach it, which seems to involve reinventing it for oneself. So, spoiler alert?

Chapter 2

Background

Above all else, do not read this paper in the order it is written, until you become concerned with circularity ([3], chapter 2). Our main technical difficulty will lie in boot-strapping the construction by loading the induction hypotheses with very special cases of the most fundamental properties of the integral, which is, after all, our goal. These properties can be understood before we check that circularity has been properly eschewed, which may seriously help motivate and clarify the careful steps taken to given the construction a proper foundation. In particular, none of the examples given are mathematically necessary for the results to come.

2.1 Basic Notations and Terminology

Post-fixed operations such as $\textsf{-Cat}$, $\textsf{-Grph}$, $\textsf{-Mod}$ are all understood to bind more tightly (before) other operations, including those written in superscript. In particular, $n\textsf{-Cat}(A, B) = (n\textsf{-Cat})(A, B)$, $n\textsf{-Cat}^* = (n\textsf{-Cat})^*$ and so forth.

I denote isomorphism by $\approx$. Diagrams used grammatically as statements commute. The symbol $\Box$ is used as a place-holder, as in $f = f(\Box)$.

The following terminology is used throughout. An *action* X of a category C is a functor $C \xrightarrow{X} \mathsf{Set}$. When X is understood, for an arrow $a \xrightarrow{f} b$ of C and an element $p \in X(a)$, I will write

$$fp := (X(f))(p),$$

calling $X(f)$ the *action of the arrow* f. For an additional arrow $b \xrightarrow{g} c$, we have $1_a p = p$ and $g(fp) = (gf)p$, and conversely, these relations make X into a functor.

Recall that a functor is called faithful to mean that it induces injections on the level of hom-sets. A *concrete* category C is a category C together with a canonized faithful action. Diagram chasing happens in this context: to show that arrows are equal, one can check that they act the same way. Every category $n\text{-}\mathsf{Cat}$ is concrete, the canonical action being on the sets of cells.

Every paper is written for an audience, and in an effort to be as inclusive as possible I have taken pains to ensure that background material is exhibitted in a greater level of detail than some readers may care to process. In these cases, I have tried to include short summaries, mostly packed full of the relevant notations, to serve as alternatives for readers already well-versed in the concepts. Thus, occasionally, I may suggest skipping to a section (like 2.3 or 2.4 for a rapid introduction to n-categories), or browsing a section to determine which other sections to skip, skim or read in detail.

Now the exposition splits. Greater detail for the terse paragraphs that follow can be found in the indicated section numbers.

(2.2) Affixing a prime to a categorical construction indicates passage to an enveloping (larger) universe of set theory. This is possible within ordinary (ZF+choice) set theory by Godel's completeness theorem.

(2.3) I write $B\text{-}\mathsf{Grph}$ for the category of B-enriched graphs, $|\Box|$ for the object-set functor from $B\text{-}\mathsf{Grph}$, and $\Gamma(a, b)$ for the hom-object in the B-graph Γ between the

objects a and b. The category of n-graphs is denoted $n\text{-}\mathsf{Grph}$.

(2.4) I denote the (canonical choice of) terminal object of a category B by $\mathbb{1}_B$. All my closed categories B will be cartesian closed, the closure adjunction being $B(a\times b,c)\xrightarrow[\approx]{\mathsf{rad}}B(a,c^b)$ with counit written $\mathsf{eval} = \mathsf{eval}^{(b)}_{a,c}$. I refer to adjuncts under this adjunction as curries, in honor of Haskell B. Curry. (This is almost common among theoretical computer scientists.) The category $B\text{-}\mathsf{Cat}$ is the category of B-enriched categories. In a B-category C, $C^{[[a]]}$ denotes the identity $\mathbb{1}_B \longrightarrow C(a,a)$ at the object a of C, and $C^{[b]}_{a,c}$ denotes the multiplication arrow

$$C(b,c)\times C(a,b)\longrightarrow C(a,c)\,.$$

(When these notations are used in the sequel I will remind the reader what they mean.) The $(n+1)$-category of n-categories is written $n\text{-}\mathsf{Cat}$. I refer to the k-cells of $n\text{-}\mathsf{Cat}$, $k>1$, as n-natural $(k-1)$-transformations. For a k-cell f of an n-category C, I write $\mathsf{dom}_m\, f$ and $\mathsf{cod}_m\, f$ for the domain and codomain m-cells, $m\le k\le n$. The k-cells g and f are composable along an m-cell when $\mathsf{cod}_m\, f = \mathsf{dom}_m\, g$, and then this composite is written $g\cdot_m f$. Arbitrary dimensional cells are displayed diagramatically with their dimension in brackets, so that

$$\begin{tikzcd} a_{[k]} \arrow[r, bend left, "f", "{[l]}"'] \arrow[r, bend right, "g"', "{[l]}"] \arrow[r, phantom, "{[j]\Downarrow x}"] & b_{[k]} \end{tikzcd}$$

displays a j-cell x with $\mathsf{dom}_l\, x = f$ and $\mathsf{dom}_l\, x = g$, where f and g are l-cells with domain and codomain k-cells a and b, respectively. Any omitted dimension (outside such a diagram) is 1, so $\cdot$ is composition along 1-cells; I reserve juxtaposition for $\cdot_0$, composition along 0-cells.

(2.5) For an n-category C, the opposite n-category obtained by reversing all 1-cells is denoted $\overline{C}$ while the n-category obtained by reversing all k-cells in dimensions

$k > 1$ is written C^*. These dualities are the important ones for my constructions, since the $(n+1)$-functor $n\text{-}\mathsf{Cat}^* \xrightarrow{\overline{\square}^*} n\text{-}\mathsf{Cat}$ (reverse all cells) plays a central role. I denote the truncation of an n-category C to dimension k as $\lfloor C \rfloor_k$ or simply as C when the context demands a k-category. Likewise, $\lceil C \rceil_k$ denotes the inflation of C to dimension k (by adding identity cells only), and when the context demands a k-category I may simply write C. I identify, in a chain of subcategories,

$$0\text{-}\mathsf{Cat} \leq 1\text{-}\mathsf{Cat} \leq 2\text{-}\mathsf{Cat} \leq \cdots \leq \mathsf{fdCat} \leq \infty\text{-}\mathsf{Cat}$$

where fdCat is the category of finite dimensional ∞-categories (the colimit of the truncation functors) and $\infty\text{-}\mathsf{Cat}$ is the category of ∞-categories (the limit of the inflation functors).

If this whole section made sense, you'll do well to skip to section 2.6.

2.2 Foundations

Since category theory tends to concern itself with the whole of a branch of mathematical endevor at a time, it runs into foundational considerations quickly. The collection of all sets cannot be a set, by the axiom of regularity, or by Russel's paradox.

The lack of proper foundations did not stop Euclid, Newton or Fourrier from producing enduring mathematics; rather, their break-throughs (arguably) motivated the development of our modern foundations (such as the theories of sets, limits and measures). The perspective of Thomas Kuhn's "The Structure of Scientific Revolutions" ([7]) seems to apply in this way to mathematical history as well.

I'm not so much concerned with discovering the true foundations of mathematics, but rather with conducting my mathematics safely away from the fundamental issues. Should set theory meet its end, whatever saves algebra, topology and analysis

should save my work, too, since I don't do any of the exotic things that define the border between competing versions of set-theory. On the other hand, since sets are 0-categories, higher category theorists such as myself can no longer simply wash our hands of this whole mess: set theory *is* (the 0-dimensional case of) higher category theory.

So the foundation I present here is not intended as the one true foundation for all mathematics, although for all I know it might be. Here is one way that someone who wants to use category theory within the scope of conventional set theory to do it. Tacitly, such of us should be willing to assume the consistency of set theory, no matter how much great logic of the last century has been devoted to this open question. (Almost any competing set-theory will do, so long as it is known to be no less consistent than Zermelo-Fraenkl and so long as it affords me the axiom of choice.)

Most pure mathematics is done using the Godel-Bernays axioms for set theory, that is, "sets" and "classes," but for categorical purposes this quickly becomes inadequate, as the collection of all classes cannot be a class. Moreover, the theories of "sets" and of "classes" are elementary equivalent in the sense that any statement (making no reference to the embedding of one model in the other) which is provable in one is provable in the other.

The solution adopted by most categorists is to let the words "set," "map," etc. play as variables. Thus a *set* is henceforth an element of some (variable) model of set theory.

We still have the problem of obtaining the existence of a suitable tower of models (the sets, classes and whatever-comes-nexts). An efficient short-cut to these results of Godel-Bernays is an inductive process. Assume that standard (ZF+choice, to be specific) set theory is consistent. Godel's completeness theorem guarantees the existence of a model, which is to say an actual set of elements (its "sets") and an

actual membership relation on it, for this consistent theory. Therefore, we have a model of set theory within set theory.

We may now apply the completeness theorem inductively within this new model, and iterate this process any finite number of times to obtain a tower of finitely many nested models. It may be counterintuitive that the "real" sets are the outermost model, but even "real" classes are born counterintuitively (at least to us mathematical Platonists). I hereby proclaim a "set," under ordinary circumstances, to be an element of the innermost model so defined, and define a "class" to be an element of the next innermost model. I will not need names for the elements of the outermost models, and so I refrain from naming them. I hereby reserve the word "collection" for an element of some unknown model of set theory.

Those wary of all this will easily be able to translate what I have done to "sets," "classes," and just plain logic, since I will only ever need the innermost three models of set theory, and the third level doesn't even need a name. But for those whose mathematical work doesn't always start with ("small") sets, it's good to know nothing in this paper really depends on that.

Notationally, I will affix a prime to the expansion of a construction based on sets to a larger model. For example, by Russell's paradox I can't have Cat in Cat, so I have to settle for Cat in Cat' in Cat'', and so on. (But I promise no triple primes or beyond.)

2.3 Globular Complexes and Enriched Graphs

Different expositions of this material are availible in [6], and [8]. However, what follows is the most direct path to chapter 3.

A (directed) *globular complex* is a cell complex (with no second-countability as-

sumed, and where recall the sets may be classes as in section 2.2) in which every k-cell ($k > 0$) is simply attached to two $(k-1)$-cells, designated its *northern* and *southern hemispheres*, which either intersect in their shared boundary, or coincide. Their shared boundary is called its *equator*, and each such cell is called a *globe*. (Despite the euphonic vowel and stress shifts, one general meaning of "globular" is "like or composed of globes" – check your favorite dictionary!)

In this perspective, to be completely rigorous, the 0-cells must all be considered attached, north and south, to the unique (-1)-cell, representing the empty set. This can be counterintuitive when thinking about the next picture, but the pictures for $k = 2$ and $k = 3$ which follow should clarify the equatorial condition imposed above. The terminology here may be considered more trustworthy than playful.

Illustrating the direction, north to south, by an arrow, at $k = 1$, a globe is an edge or arrow:

$$\cdot \longrightarrow \cdot$$

at $k = 2$ it is a disk directed as in:

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot \end{tikzcd}$$

and at $k = 3$ we can think of the usual globe mapping the Earth in the standard terminology, although we have to direct the equator, say from 0 degrees longitude to 180 degrees in each direction, and the hemispheres' surfaces, say from the eastern equator to the western, over the poles, finally directing the interior of the planet, from the northern hemisphere to the southern.

Writing $\mathsf{dom}\, g$ and $\mathsf{cod}\, g$ for the northern, or *domain*, and southern, or *codomain* hemispheres, respectively, of a globe g in a globular complex Γ, and $\Gamma^{(k)}$ for the collection of k-globes in Γ we arrive at a combinatorialization of Γ as a collection of maps:

$$\Gamma^{(0)} \underset{\mathsf{cod}}{\overset{\mathsf{dom}}{\leftleftarrows}} \Gamma^{(1)} \underset{\mathsf{cod}}{\overset{\mathsf{dom}}{\leftleftarrows}} \Gamma^{(2)} \underset{\mathsf{cod}}{\overset{\mathsf{dom}}{\leftleftarrows}} \cdots$$

satisfying the equatorial relations that $\mathsf{dom}\,\mathsf{dom} = \mathsf{dom}\,\mathsf{cod}$ and $\mathsf{cod}\,\mathsf{dom} = \mathsf{cod}\,\mathsf{cod}$. (This hints at a homological boundary operator given by $\mathsf{dom} - \mathsf{cod}$, after linearization, which I will have no use for.)

As such, these (combinatorial directed) globular complexes, or ∞-*graphs*, form a category $\infty\text{-}\mathsf{Grph}$ as the category of functors from J to Set, where J is the category presented by the generators

$$0 \underset{c_0}{\overset{d_0}{\leftleftarrows}} 1 \underset{c_1}{\overset{d_1}{\leftleftarrows}} 2 \underset{c_2}{\overset{d_2}{\leftleftarrows}} \cdots$$

and "equatorial" relations $d_i d_{i+1} = d_i c_{i+1}$ and $c_i d_{i+1} = c_i c_{i+1}$, $i = 0, 1, \ldots$. Thus a morphism $\Gamma \xrightarrow{\varphi} \Delta$ in $\infty\text{-}\mathsf{Grph}$ consists of a sequence of maps $\langle \Gamma^{(k)} \xrightarrow{\varphi^{(k)}} \Delta^{(k)} \rangle_k$ such that,

$$\begin{tikzcd} \Gamma^{(k)} \arrow[d, "\varphi^{(k)}"'] & \Gamma^{(k+1)} \arrow[l, "\mathsf{dom}"'] \arrow[d, "\varphi^{(k+1)}"] \\ \Delta^{(k)} & \Delta^{(k+1)} \arrow[l, "\mathsf{dom}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Gamma^{(k)} \arrow[d, "\varphi^{(k)}"'] & \Gamma^{(k+1)} \arrow[l, "\mathsf{cod}"'] \arrow[d, "\varphi^{(k+1)}"] \\ \Delta^{(k)} & \Delta^{(k+1)} \arrow[l, "\mathsf{cod}"] \end{tikzcd}$$

for $k = 0, 1, \ldots$. We may assume all the sets $\Gamma^{(k)}$ are mutually disjoint, for such ∞-graphs induce a dense full subcategory.

These notations won't play much of a role in the sequel, but it is important to have the right visual image of an ∞-graph. More important is the following alternate formulation of the finite dimensional case, which, although perhaps less immediately vivid, leads to much more pleasant bookkeeping:

For any base category B, the category $B\text{-}\mathsf{Grph}$ of B-*enriched graphs* (or B-*graphs* for short) is defined as follows. The objects of $B\text{-}\mathsf{Grph}$ are ordered pairs $\Gamma = (|\Gamma|, \langle \Gamma(a,b) \rangle_{a,b \in |\Gamma|})$, where $|\Gamma|$ is a set, called the set of vertices or *objects*, and each $\Gamma(a,b)$ is an object of B, called the quiver or *hom-object* from a to b in Γ. The arrows

$\Gamma \xrightarrow{\varphi} \Delta$ in B-Grph are ordered pairs $\varphi = (|\varphi|, \langle \varphi_{a,b} \rangle_{a,b \in |\Gamma|})$ where $|\Gamma| \xrightarrow{|\varphi|} |\Delta|$ is a map, called the object map for φ and each $\varphi_{a,b}$ is an arrow $\Gamma(a,b) \xrightarrow{\varphi_{a,b}} \Delta(|\varphi|a, |\varphi|b)$ in B, called the *hom-arrow* for φ from a to b. The composition and identities are reasonably obvious.

This relates to the construction of ∞-Grph as follows. Define 0-$\mathsf{Grph} = \mathsf{Set}$ to be the category of sets, and then inductively define

$$(n+1)\text{-}\mathsf{Grph} := (n\text{-}\mathsf{Grph})\text{-}\mathsf{Grph},$$

$n = 0, 1, \ldots$, calling each n-Grph the category of n-graphs. Each n-Grph is equivalent to the full subcategory of ∞-Grph of all globular complexes of dimension at most n.

To clarify, in detail, a 2-graph Γ in 2-Grph has:

- $|\Gamma|$, its set of 0-cells
- $\Gamma(a,b)$, a 1-graph of cells going from the 0-cell a to the 0-cell b in Γ, for every a and b, containing
- $|\Gamma(a,b)|$, the set of 1-cells from a to b in Γ and, for every $f, g \in |\Gamma(a,b)|$,
- $[\Gamma(a,b)](f,g)$, the set of 2-cells from f to g, which must both run from a to b by an equatorial relation.

I will associate to the left in expressions like the last one, i.e.

$$\Gamma(a_i, b_i)_{i=0}^{n} = \Gamma(a_0, b_0)(a_1, b_1) \cdots (a_n, b_n) = [\cdots [[\Gamma(a_0, b_0)](a_1, b_1)] \cdots](a_n, b_n)$$

refers to the k-graph of cells attached in Γ from a_n to b_n, which both run from a_{n-1} to b_{n-1}, …, which both run from a_0 to b_0.

2.4 Enriched and Higher Categories

Again, different expositions of this material are availible in [6], and [8]. However, what follows is again the most direct path to chapter 3. (This remark repeats that heading section 2.3.)

Higher categories have globular complexes as their underlying structure, but they are most easily defined and manipulated as certain enriched categories, more akin to the definition of enriched graphs.

Let B be a base category which is complete and closed. Recall these last two adjectives:

The completeness of B means that every functor $J \xrightarrow{T} B$ has a limit $\mathsf{Lim}\, T$ in B. Writing B^J for the category of functors from J to B, this means every diagonal (constant functor forming) functor $B \xrightarrow{\Delta} B^J$ has a right adjoint $B^J \xrightarrow{\mathsf{Lim}} B$ so that $B^J(\Delta b, T) \approx B(b, \mathsf{Lim}\, T)$ naturally in b and T. In particular, B has all direct products, including the empty product or terminal object $\mathbb{1}_B$. In Set, $\mathbb{1}_{\mathsf{Set}} = \{()\}$ is the singleton set of the empty tuple (the canonical direct product of zero sets), or choose your favorite singleton. (More detailed background can be found in section 3.7.3.)

The assumption that B is closed means that for every object b of B, the functor $B \xrightarrow{(\times b)} B$ (direct product by b) has a right adjoint $B \xrightarrow{\square^b} B$ so that $B(a \times b, c) \xrightarrow[\approx]{\mathsf{rad}} B(a, c^b)$, naturally in a and c. At $a = c^b$, writing

$$\mathsf{eval} = \mathsf{rad}^{-1}\, 1_{c^b},$$

this is expressed by the universal property that every arrow $a \times b \xrightarrow{f} c$ of B factors as $a \times b \xrightarrow{f' \times b} c^b \times b \xrightarrow{\mathsf{eval}} c$ for a unique arrow $a \xrightarrow{f'} c^b$ of B, namely $f' = \mathsf{rad}\, f$, the right adjunct to f. In honor of Haskell B. Curry, I refer to f and f' as *"curries"* of one another. (This terminology is common among certain computer-scientific mathematicians.) In Set, c^b is the set of functions from b to c.

We will construct the category B-Cat of *B-(enriched) categories* and *B-functors*. As you read the following, it may help to think of B as the category Set of sets, in which case, Set-Cat will be (equivalent to) the category Cat of categories. Enrichment can be done in a fantastically more general setting (see [6]), but this would distract from and delay the introduction of higher categories.

The category B-Cat is defined as follows. An object C of B-Cat is a quadruple

$$C = (|C|, \langle C(a,b)\rangle_{a,b\in|C|}, \langle C^{[[a]]}\rangle_{a\in|C|}, \langle C^{[b]}_{a,c}\rangle_{a,b,c\in|C|})$$

where $(|C|, \langle C(a,b)\rangle_{a,b})$ is the underlying B-graph of C, each $C^{[[a]]}$ is an arrow $\mathbb{1}_B \longrightarrow C(a,a)$ in B, called the identity at a, and each $C^{[b]}_{a,c}$ is an arrow $C(b,c)\times C(a,b) \longrightarrow C(a,c)$ in B, called the composition arrow from a to c via b, subject to the following axioms. (Don't worry – I'll remind you of these last two notations when they are used in the sequel.)

Associativity: For $a,b,c,d \in |C|$ we have in B:

$$\begin{tikzcd}
C(c,d)\times C(b,c)\times C(a,b) \arrow[r, "C(c,d)\times C^{[b]}_{a,c}"] \arrow[d, "C^{[c]}_{b,d}\times C(a,b)"'] & C(c,d)\times C(a,c) \arrow[d, "C^{[c]}_{a,d}"] \\
C(b,d)\times C(a,b) \arrow[r, "C^{[b]}_{a,d}"'] & C(a,d)
\end{tikzcd}$$

Identity: For $a,b \in |C|$ we have in B both:

$$\begin{tikzcd}
\mathbb{1}_B\times C(a,b) \arrow[r, "C^{[[b]]}\times C(a,b)"] \arrow[dr, "\approx"'] & C(b,b)\times C(a,b) \arrow[d, "C^{[b]}_{a,b}"] \\
& C(a,b)
\end{tikzcd}$$

and

$$\begin{tikzcd}
C(a,b)\times \mathbb{1}_B \arrow[r, "C(a,b)\times C^{[[a]]}"] \arrow[dr, "\approx"'] & C(b,b)\times C(a,b) \arrow[d, "C^{[a]}_{a,b}"] \\
& C(a,b)
\end{tikzcd}$$

where the arrows labelled "$\approx$" are the canonical isomorphisms.

Whew! That's a B-category. In the case $B = \mathsf{Set}$ what we have defined is equivalent to the usual definition of a category. (Although it lacks the disjointness axiom that $C(a,b) \cap C(a',b') = \emptyset$ unless $(a,b) = (a',b')$, the Set-categories satisfying this condition form a dense, full, and therefore equivalent subcategory of Set-Cat.)

It is common enough for the compositions and identities in a B-category to be denoted by single letters like m, μ, i and ι, suppressing their abundant subscripts. In applications, experience has shown this can waste effort, since many applications will already have used these key letters. Of course, many readers will enjoy a reminder, in the surrounding text, of the meanings of these complicated-looking notations, but at least those who use this paper and its notations will be freed of trying to find new letters to represent these core concepts. (Metric space enthusiasts, why not write $X_\epsilon(p)$ for the ϵ-ball around p? That X is lazy – it can handle the extra work! Give it a Minkowski sausage.)

The arrows $D \xrightarrow{F} C$ of B-Cat, called *B-functors*, are B-graph morphisms satisfying the two conditions:

F preserves composition: For $a, b, c \in |D|$, we have in B,

$$\begin{tikzcd}
D(b,c) \times D(a,b) \arrow[r, "D_{a,c}^{[b]}"] \arrow[d, "F_{b,c} \times F_{a,b}"'] & D(a,c) \arrow[d, "F_{a,c}"] \\
C(|F|b, |F|c) \times C(|F|a, |F|b) \arrow[r, "C_{|F|a,|F|c}^{[|F|b]}"'] & C(|F|a, |F|c)
\end{tikzcd}$$

F preserves identities: We have, in B,

$$\begin{tikzcd}
\mathbb{1}_B \arrow[r, "D^{[[a]]}"] \arrow[d, "="'] & D(a,a) \arrow[d, "F_{a,a}"] \\
\mathbb{1}_B \arrow[r, "C^{[[|F|a]]}"'] & C(|F|a, |F|a)
\end{tikzcd}$$

for every $a \in |D|$.

It is customary to write simply $Fa = |F|a$ in this context, as the absolute value bars become a needless encumberance.

Now, B-Cat is complete and closed. Here are a few of the details; for all of them, see [6], or just rewrite [9] in terms of diagrams – after all, this is what enrichment really does. Let C and D be B-categories. The B-category $C \times D$ has $|C \times D| = |C| \times |D|$ and $[C \times D][(c,d),(c',d')] = C(c,c') \times D(d,d')$. Other limits in B-Cat have similar descriptions in terms of limits in Set and limits in B. The B-category C^D has $|C^D|$ the set of B-functors from D to C, and for $S, T \in |C^D|$, $C^D(S,T)$ is an equalizer of a pair of arrows in B of the form

$$\Pi_{d \in |D|} C(Sd, Td) \rightrightarrows \Pi_{d,e \in |D|} C(Sd, Te)^{D(e,d)}$$

where these arrows have components currying

$$C(Se,Te) \times D(d,e) \xrightarrow{C(Se,Te) \times S_{d,e}} C(Se,Te) \times C(Sd,Se) \xrightarrow{C^{[Se]}_{Sd,Te}} C(Sd,Te)$$

and

$$D(d,e) \times C(Sd,Td) \xrightarrow{T_{d,e} \times C(Sd,Td)} C(Td,Te) \times C(Sd,Td) \xrightarrow{C^{[Td]}_{Sd,Te}} C(Sd,Te)$$

emulating the condition, for ordinary categories C and D that $S \xrightarrow{X} T$ in C^D is natural at $d \xrightarrow{f} e$ in D, i.e. in C,

$$\begin{array}{ccc} Sd & \xrightarrow{Xd} & Td \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ Se & \xrightarrow[Xe]{} & Te \end{array}$$

(It is a classical result that all limits can be obtained from just products and equalizers. Recall, equalizers are the poor-man's subobjects.)

Now, finally, finally, the application. Define the category of 0-categories as $0\text{-}\mathsf{Cat} = \mathsf{Set}$ and inductively define the category of $(n+1)$-categories as $(n+1)\text{-}\mathsf{Cat} = (n\text{-}\mathsf{Cat})\text{-}\mathsf{Cat}$, $n = 0, 1, \ldots$. A $n - \text{-}\mathsf{Cat}$-functor is called an $(n+1)$-functor, and higher categories and functors are n-categories and n-functors for unspecified n.

But what does all this amount to? An n-category C is supported by an n-graph, i.e. an n-dimensional (directed) globular complex. Every k-globe a of C, $k < n$ has an identity $(k+1)$-globe 1_a attached from a to a.

There is also a system of compositions; to describe them I need the notion of the m-domain and m-codomain of a k-cell a, $0 \leq m \leq k \leq n$, denoted $\mathsf{dom}_m\, a$ and $\mathsf{cod}_m\, a$ respectively. If $m = k$ then $\mathsf{dom}_m\, a = \mathsf{cod}_m\, a = a$. Otherwise $m < k$ and we define $\mathsf{dom}_m\, a = \mathsf{dom}_m \,\mathsf{dom}\, a$ and $\mathsf{cod}_m\, a = \mathsf{cod}_m \,\mathsf{cod}\, a$. Thus $\mathsf{dom}_m\, a$ and $\mathsf{cod}_m\, a$ are either a itself, or a northern/southern hemisphere, respectively, of dimension m along the chain of attachments for a.

Now, for $0 \leq m \leq k \leq n$, any two k-cells f and g of C with $\mathsf{dom}_m\, g = \mathsf{cod}_m\, f$ have a composition along this common m-cell, denoted $g \cdot_m f$, and both $\mathsf{dom}_m(g \cdot_m f) = \mathsf{dom}_m\, f$ and $\mathsf{cod}_m(g \cdot_m f) = \mathsf{cod}_m\, g$. This is represented diagrammatically by

$$a_{[m]} \xrightarrow[{[k]}]{f} b_{[m]} \xrightarrow[{[k]}]{g} c_{[m]}$$

where $a = \mathsf{dom}_m\, f$, $b = \mathsf{dom}_m\, g$ and $c = \mathsf{cod}_m\, g$. The cells in such a diagram which may not be pictured in their true dimension are labelled by their true dimension, enclosed in brackets. (So when $m = 0$ or $k = 1$ we may omit these labels.)

In the case $a_{[m]} \xrightarrow[{[k]}]{f} b_{[m]} \xrightarrow[{[j]}]{g} c_{[m]}$, we identify $f = 1_f = 1_{1_f} = \cdots$ and $g = 1_g = 1_{1_g} = \cdots$ so that $g \cdot_m f$ is defined as a $\mathsf{max}\{k, j\}$-cell. This very important convention is at play whenever the dimension of a cell is lower than the context demands.

The associative law guarantees the nonambiguity of compositions such as

$$a_{[m]} \xrightarrow[{[k]}]{f} b_{[m]} \xrightarrow[{[k]}]{g} c_{[m]} \xrightarrow[{[k]}]{h} d_{[m]}\,.$$

The identity laws assert that, for $a_{[m]} \xrightarrow[{[k]}]{f} b_{[m]}$, $f \cdot_m 1_a = f = 1_b \cdot_m f$.

Finally, for $0 \leq m \leq k \leq l \leq n$, the $(n-m)$-functoriality of the composition along m-cells gives a *dimensional interchange law* asserting the nonambiguity of compositions such as

$$\begin{tikzcd}
a_{[m]} \arrow[r, "f", "{[l]}"'] \arrow[d, "="'] & b_{[m]} \arrow[r, "g", "{[l]}"'] \arrow[d, "="'] & c_{[m]} \arrow[d, "="] \\
a_{[m]} \arrow[r, "f'", "{[l]}"'] \arrow[d, "="'] \arrow[u, phantom, "{[k]}\Downarrow\varphi", xshift=40] & b_{[m]} \arrow[r, "g'", "{[l]}"'] \arrow[d, "="'] \arrow[u, phantom, "{[k]}\Downarrow\gamma", xshift=40] & c_{[m]} \arrow[d, "="] \\
a_{[m]} \arrow[r, "f''", "{[l]}"'] \arrow[u, phantom, "{[k]}\Downarrow\varphi'", xshift=40] & b_{[m]} \arrow[r, "g''", "{[l]}"'] \arrow[u, phantom, "{[k]}\Downarrow\gamma'", xshift=40] & c_{[m]}
\end{tikzcd}$$

which is to say that, when both sides are defined,

$$(\gamma' \cdot_l \gamma) \cdot_m (\varphi' \cdot_l \varphi) = (\gamma' \cdot_m \varphi') \cdot_l (\gamma \cdot_m \varphi).$$

If you collapse the vertical arrows to points, the picture resembles a butterfly's four wings, so these laws are also commonly known as *butterfly laws*.

In low dimensions, $\cdot_1$ is simply written $\cdot$, and $\cdot_0$ is written as juxtaposition. (Even though I won't have the pleasure of using it, note how sensible it would be to write $(\otimes) = (\cdot_{-1})$.)

Conversely, this pile of compositions, identities, associative, identity and butterfly laws does define an n-category, and so it helps inform our intuition of what one is. On the other hand, the number of butterfly laws grows cubically with n since there is one for every choice of 3 dimensions. Most of the time, mixing this globular perspective with the enriched perspective will yield the most efficient proofs.

Example 2.4.1 (Ordered Categories)**.**

For reasons soon to become clear, category theorists often study *preordered* sets, i. e. those bearing reflexive and transitive relations. Classical order theory emerges

then from basic facts about adjunctions and skeletons.

An *ordered category* C is one in which every hom-set $C(a,b)$ bears a preorder relation, and in which every composition $C(b,c) \times C(a,b) \longrightarrow C(a,c)$ is order-preserving. They may equally well be defined as those 2-categories in which there is at most one 2-cell from a given 1-cell to another 1-cell.

Example 2.4.2 (Rewriting Systems)**.**

A *rewriting system* in monoid theory is an order relation on a free monoid. Thinking of this monoid as a category with exactly one object, $\star$, we obtain an ordered category, which is an example of a 2-category. To illustrate, here is a deduction that under the rules $ba \to ab$ and $aa \to a$ we have $aba \to ab$, pictured as a composition of 2-cells:

$$\begin{tikzcd}
 & \star \arrow[rr, "b"] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow"] & \arrow[d, phantom, "\Downarrow"] & \star \arrow[dr, "a"] & \\
\star \arrow[ur, "a"] \arrow[rr, "a"'] & & \star \arrow[rr, "b"'] & & \star
\end{tikzcd}$$

Example 2.4.3 (Topological Spaces)**.**

The 2-category Top of topological spaces, continuous maps, and relative homotopy classes of map homotopies is given as follows. The 0-cells of Top are the topological spaces, and the 1-cells of Top are the continuous maps. The composition of 1-cells along 0-cells is the usual composition of maps along spaces.

Let $I := [0,1]$ be the usual subspace of the real line.

The 2-cells $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow{[H]}"] & Y \end{tikzcd}$ are relative homotopy classes of homotopies $I \times X \xrightarrow{H} Y$ with $H_0 = f$ and $H_1 = g$. Specifically, $[H] = [K]$ precisely when there is a homotopy

$I \times I \times X \xrightarrow{M} Y$ with $M_0 = H$, $M_1 = K$ and for every $t \in I$ both $(M_t)_0 = f$ and $(M_t)_1 = g$.

For any map f, the identity 2-cell at f is $1_f = [L]$ where L is the homotopy with $L_t = f$ for every $t \in I$.

For 2-cells that meet along a map, $f \xrightarrow{[H]} g \xrightarrow{[K]} h$ in $\mathsf{Top}(X, Y)$, the composite along g is $[K] \cdot [H] = [L]$ where L is the homotopy satisfying

$$L_t = \begin{cases} H_{2t} & \text{for } 0 \leq t \leq 1/2 \\ K_{2t+1} & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

for every $t \in I$.

For 2-cells that meet along a space, as in

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow[H]"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow[K]"] & Z \end{tikzcd}$$

the composition $[K][H] = [L]$ is represented by the homotopy L with $L_t = K_t H_t$ for every $t \in I$.

The proof that composition along 1-cells is defined making each $\mathsf{Top}(X, Y)$ into a category is virtually identical to the proof that the fundamental groupoid $\Pi_1 Y$ is a category. Indeed, if we take X to be the one-point space $\mathbb{1}_{\mathsf{Top}}$, then the maps $\mathbb{1}_{\mathsf{Top}} \longrightarrow Y$ are the points of Y, and the 2-cells connecting them are the path-homotopy classes of paths. In fact, $\Pi_1 Y \approx \mathsf{Top}(\mathbb{1}_{\mathsf{Top}}, Y)$ 2-naturally in Y. Every $\mathsf{Top}(X, Y)$ is a groupoid, and fixing either X or Y gives a 2-functor, hence an homotopy invariant.

One must also verify well-definedness, the associative and identity laws for the composition of 2-cells along 0-cells, as well as a single butterfly law, in order to conclude this is an example 2-category.

Example 2.4.4 (n-Cat)**.**

The most protean example of an n-category for $n > 0$ is $(n-1)\text{-}\mathsf{Cat}$ itself. When the base category B is closed, B itself is a B-category, taking $B(b,c) := c^b$ for the hom-objects. (This notation is slightly ambiguous, but canonical actions are hardly ever written.) Since every $n\text{-}\mathsf{Cat}$ is closed, every $n\text{-}\mathsf{Cat}$ is thus a $n\text{-}\mathsf{Cat}$-category, i.e. an $(n+1)$-category. Sometimes understanding starts with believing, and this is so miraculous it may help to think about low dimensions: $0\text{-}\mathsf{Cat} = \mathsf{Set}$ is a 1-category, and $1\text{-}\mathsf{Cat} \approx \mathsf{Cat}$ is a 2-category. (The 2-cells of Cat are called natural transformations.)

We have already met the 0- and 1-cells of $n\text{-}\mathsf{Cat}$. For $2 \leq k \leq n+1$, the k-cells $\begin{tikzcd} D \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "{[k]}\Downarrow X"] & C \end{tikzcd}$ of $n\text{-}\mathsf{Cat}$, which I shall call *n-natural $(k-1)$-transformations*, are families of $(k-1)$-cells of C: $X = \langle Sd \xrightarrow[{[k-1]}]{Xd} Td \rangle_{d\in|D|}$ such that for any $d \xrightarrow[{[l]}]{f} c$ in D, we have, in C,

$$\begin{tikzcd} Sd \arrow[r, "Xd", "{[k-1]}"'] \arrow[d, "Sf"', "{[l]}"] & Td \arrow[d, "Tf", "{[l]}"'] \\ Se \arrow[r, "{[k-1]}", "Xe"'] & Te \end{tikzcd}$$

which is called the n-naturality of X at f.

This terminology is in keeping with the standard practice of making any omitted dimension the classical dimension, one. In practice, an n-natural 3-transformation is more commonly called a modification, while 4-transformations get called perturbations, but it behooves us to have a name which is linguistically regular in the dimension.

At $k = 2$ we have $\mathsf{dom}\, X = S$ and $\mathsf{cod}\, X = T$ and for $k > 2$ we have $\mathsf{dom}\, X = \langle \mathsf{dom}\, Xd \rangle_{d\in|D|}$ and $\mathsf{cod}\, X = \langle \mathsf{cod}\, Xd \rangle_{d\in|D|}$. Note, if $k > 2$, then by a butterfly law it suffices to check the square above at $l = 1$.

Defining $Xf = (Xe)(Sf) = (Tf)(Xd)$ makes evaluation into an n-functor

$$n\text{-}\mathsf{Cat}(D,C) \times D \longrightarrow C\,.$$

Composition of k-cells along 0-cells is defined as in Cat: for

$$\begin{tikzcd} D \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "{[k]\Downarrow X}"] & C \arrow[r, bend left, "U"] \arrow[r, bend right, "V"'] \arrow[r, phantom, "{[k]\Downarrow Y}"] & B \end{tikzcd}$$

and $d \in |D|$, we have $(YX)d = Y(Xd) = (VXd)(YSd) = (YTd)(UXd)$

For arbitrary k-cells X and Y, $k > 1$ of $n\text{-}\mathsf{Cat}$, composition along l-cells, $l > 0$, (when defined) is given by $(Y \cdot_l X)d = (Yd) \cdot_{l-1} (Xd)$, for all $d \in |D|$.

Finally, the identity $(k+1)$-cell 1_S at a k-cell S, $k > 0$ is given on $d \in |\,\mathsf{dom}_0\, S|$ by $(1_S)d = 1_{Sd}$.

2.5 Important Constructions

For any n-category C, the n-category $\overline{C}$ is the same n-category, but with all 1-cells and the composition along 0-cells reversed. In the case $n = 1$, this is the usual opposite category. The n-category C^* is obtained by reversing all cells in dimensions above 1 and all compositions in dimensions above 0. This other kind of duality plays a role in my work, since reversing all cells and compositions defines an $(n+1)$-functor $n\text{-}\mathsf{Cat}^* \xrightarrow{\overline{\square}^*} n\text{-}\mathsf{Cat}$, which I will use frequently. (A wise man [1] once told me that the "Stars and bars" tell you what kind of math you're reading; now we, and in particular you, know.)

For $k \leq n$ and C an n-category, $\lfloor C \rfloor_k$ denotes the truncation of C to a k-category by discarding all cells above dimension k. When the context demands a k-category, I may write simply C for $\lfloor C \rfloor_k$. The limit of these truncation functors is the category $\infty\text{-}\mathsf{Cat}$ of ∞-categories.

For $k \geq n$ and C an n-category, $\lceil C \rceil_k$ denotes the inflation of C to a k-category by adding only identity cells above dimension n and extending the compositions to

satisfy the identity laws in the only way one can. Again, when the context demands a k-category, I may simply write C for $\lceil C \rceil_k$. The colimit of these inflation functors is the category fdCat of finite-dimensional ∞-categories.

Since $\lfloor \lceil C \rceil_n \rfloor_k = C$ for $k \leq n$ and any n-category C, we may identify each $n\,\text{-}\mathsf{Cat}$ as a subcategory of fdCat producing a tower of subcategories:

$$0\,\text{-}\mathsf{Cat} \leq 1\,\text{-}\mathsf{Cat} \leq 2\,\text{-}\mathsf{Cat} \leq \cdots \leq \mathsf{fdCat} \leq \infty\,\text{-}\mathsf{Cat}$$

I adopt this perspective throughout.

2.6 Miscellaneous Notations

When it is convenient for an n-functor or n-transformation F to act on (apply to) a subscript (rather than by juxtaposition), I will display it, at least once, as $F_\square$.

When A and X are sets, an A-tuple f of elements of X is an element $f \in X^A$, which is to say a function. One sees f written variously as

$$f = \langle fa \rangle_{a \in A} = (fa)_{a \in A} = (A \ni a \mapsto fa).$$

I shall prefer and extend this first notation, to n-functors and n-natural k-transformations as follows.

When Fa is an expression that yields a k-cell of an n-category B in an n-functorial manner under the substitution of a k-cell a of an n-category A, I will write this n-functor as $F = \langle Fa \rangle_{a \in A}$. For example, for an n-functor $T_\square$ from an index n-category J to an n-category C, it is customary to write $\mathsf{Lim}_{i \in J}\, T_i$ for $\mathsf{Lim}\, T = \mathsf{Lim} \langle T_i \rangle_{i \in J}$ – these notations make this notation (and others like it, which we will use) easier to explain and understand.

Likewise, when Gb is an expression that yields a k-cell of an n-category B in an n-functorial manner under substitution of a k-cell b of the n-category $\overline{A}$, I will write this (contravariant) n-functor as $G = \langle Gb \rangle^{b \in A}$.

Given an expression X_a that yields a k-cell of an n-category B in an n-natural manner under the substitution of a 0-cell a of an n-category A, this k-transformation will be denoted $X_\Box = \langle X_a \rangle_{a \in A}$, which, except for the presence of A rather than $|A|$, is already our notation.

Finally, except in diagrams and except in the case of dom and cod, any omitted dimension is 1. E.g. $\lfloor C \rfloor = \lfloor C \rfloor_1$ is a (1-)category, $\mathsf{Cat} = 1\text{-}\mathsf{Cat}$, functors are 1-functors, natural transformations are 1-natural 1-transformations, and so on.

Chapter 3

Strict Higher Grothendieck Integrals

We now embark upon the main construction. It is rather lengthy, even after trimming it down to the bare minimum, and quite abstract, so I have included examples and low-dimensional details along the way in order to illuminate it as best I can.

The construction is an induction on the dimension n. There are two integrals, one of a covariant n-functor, the other for a contravariant one. In each case, the cells of the integral n-category are certain ordered pairs. The hom-$(n-1)$-categories for each type of integral are integrals of the opposite type, making it useful to develop both side by side. In addition, we need a few properties of each integral: a hint of their functorialities and a shadow of their continuities.

In order to make the notation more palatable, I hereby coin the following notations, which the next section will clarify. (The angle brackets refer to the notations of the last section.)

- $\int_{a \in A} U_a := \int_A \langle U_a \rangle_{a \in A} = \int_A U_\square$

- $\int^{b\in B} V_b := \int^B \langle V_b \rangle^{b\in B} = \int^B V_\Box$
- $\int_{(a,b)\in A\times B} U_a \times V_b := \int_{c\in A\times B} U_{\mathsf{proj}\, c} \times V_{\mathsf{proj}'\, c}$
- $\int^{(a,b)\in A\times B} U_a \times V_b := \int^{c\in A\times B} U_{\mathsf{proj}\, c} \times V_{\mathsf{proj}'\, c}$

Here, $A \xleftarrow{\mathsf{proj}} A \times B \xrightarrow{\mathsf{proj}'} B$ is the product diagram for A and B, which here denote n-categories.

3.1 The Structure of the Induction

There are three steps for each n:

1. **Integration of n-functors:** From an n-functor $A \xrightarrow{U_\Box} n\text{-}\mathsf{Cat}$ with A in $n\text{-}\mathsf{Cat}$ we obtain $\int_A U$ in $n\text{-}\mathsf{Cat}$. The cells of $\int_A U$ are certain ordered pairs, and, when defined,

$$(g,q)(f,p) = (gf, q(U_{\mathsf{dom}_1\, g}\, p))$$

for cells (g,q) and (f,p) of $\int_A U$. Likewise from an n-functor $\overline{B} \xrightarrow{V_\Box} n\text{-}\mathsf{Cat}$ with B in $n\text{-}\mathsf{Cat}$ we obtain $\int^B V$ in $n\text{-}\mathsf{Cat}$. Again the cells of $\int^B V$ are certain ordered pairs, and, when defined,

$$(g,q)(f,p) = (gf, (V_{\mathsf{cod}_1\, f}\, q)p)$$

for cells (g,q) and (f,p) of $\int^B V$.

2. **Mapping Integrals:** From an n-natural transformation in $(n+1)\text{-}\mathsf{Cat}'$

$$\begin{tikzcd} A \arrow[r, "U_\Box"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow P_\Box"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ B \arrow[r, "V_\Box"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

with $A \xrightarrow{F} B$ in n-Cat, we construct the n-functor

$$\int_A U \xrightarrow{\int_F P} \int_B V \text{ , also written } \int_{a\in A} U_a \xrightarrow{\int_{b=Fa} P_a} \int_{b\in B} V_b \text{ .}$$

On cells (f,p) of $\int_A U$,

$$[\int_F P](f,p) = (Ff, P_{\mathsf{cod}_0 f}\, p).$$

Likewise, from an n-natural transformation in $(n+1)$-Cat'

$$\begin{tikzcd} \overline{C} \arrow[r, "U_\square"] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\Downarrow Q_\square"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ \overline{D} \arrow[r, "V_\square"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

with $C \xrightarrow{G} D$ in n-Cat, we construct the n-functor

$$\int^C U \xrightarrow{\int^G Q} \int^D V \text{ , also written } \int^{c\in C} U_c \xrightarrow{\int^{d=Gc} Q_c} \int^{d\in D} V_d \text{ .}$$

On cells (g,q) of $\int^C U$,

$$[\int^G Q](g,q) = (Gg, Q_{\mathsf{dom}_0 g}\, q).$$

3. Products of Integrals: Given n-functors $A \xrightarrow{U_\square} n$-$\mathsf{Cat}$ and $B \xrightarrow{V_\square} n$-$\mathsf{Cat}$, with both A and B in n-Cat, we get an isomorphism of n-categories

$$[\int_{a\in A} U_a] \times [\int_{b\in B} V_b] \xrightarrow[\approx]{\sigma} \int_{(a,b)\in A\times B} U_a \times V_b \text{ ,}$$

while from n-functors $\overline{A} \xrightarrow{U_\square} n$-$\mathsf{Cat}$ and $\overline{B} \xrightarrow{V_\square} n$-$\mathsf{Cat}$, with both A and B in n-Cat, we get an isomorphism of n-categories

$$[\int^{a\in A} U_a] \times [\int^{b\in B} V_b] \xrightarrow[\approx]{\sigma} \int^{(a,b)\in A\times B} U_a \times V_b \text{ .}$$

In both cases, on cells, $\sigma((a,u),(b,v)) = ((a,b),(u,v))$.

Before implementing this induction, a few remarks are in order. The base case $n = 0$ is rather special – it is the only place where the references to $n+1$ above actually matter. In step 2, since A and C are n-categories, P and Q are only n-natural, so for $n > 0$, P and Q lie in $n\text{-}\mathsf{Cat} \leq (n+1)\text{-}\mathsf{Cat}$, but at $n = 0$, P and Q must lie in $\mathsf{Cat} = 1\text{-}\mathsf{Cat}$ since $0\text{-}\mathsf{Cat}$ has no (nonidentity) 2-cells, being only a 1-category.

Nonetheless, this phrasing of the inductive hypothesis allows $n = 0$ to be treated like any other n. A less efficient approach would be to base the induction at $n = 1$ using the results of 3.3.1 and 3.4.1, but this duplicates the work done in 3.3, which is still necessary in full generality. Therefore, I have based our induction at $n = 0$, despite this subtlety.

The other remark is that the objects asserted to lie in $n\text{-}\mathsf{Cat}$ exist within the innermost ("small") model of set-theory, since $n\text{-}\mathsf{Cat}$ is not decorated with a prime (see 1.2). In particular, when A and every U_a lie in the innermost model, so does $\int_A U$ and likewise for contravariant integrals.

3.2 The Case $n = 0$

Let $n = 0$.

To integrate a functor $A \xrightarrow{U_\square} 0\text{-}\mathsf{Cat}$ with A in $0\text{-}\mathsf{Cat}$, that is to say, a family $\langle U_a \rangle_{a \in A}$ of sets, we put

$$\int_{a \in A} U_a = \int^{a \in A} U_a = \{(a,u) | a \in A, u \in U_a\}.$$

This makes sense since $A = \overline{A}$. The composition checks since the only cells above dimension 0 are identities.

So, the integral of a 0-category-valued 0-functor is a particular construction of the disjoint union of the image 0-categories.

To integrate the natural transformation in Cat,

$$\begin{tikzcd} A \arrow[r, "U_\square"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow P_\square"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ B \arrow[r, "V_\square"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

with $A \xrightarrow{F} B$ in $0\text{-}\mathsf{Cat}$, which is to say simply a family of maps $P = \langle U_a \xrightarrow{P_a} V_{Fa} \rangle_{a \in A}$, put

$$[\textstyle\int_F P](a,u) = [\int^F P](a,u) = (Fa, P_a u)$$

for every $(a,u) \in \int_A U$. This makes sense since $F = \overline{F}$, $Fa \in B$ and $P_a u \in V_{Fa}$, and it is the required function.

Finally, for functors in Cat, $A \xrightarrow{U_\square} 0\text{-}\mathsf{Cat}$ and $B \xrightarrow{V_\square} 0\text{-}\mathsf{Cat}$, with both A and B in $0\text{-}\mathsf{Cat}$, note that the following conditions are equivalent:

- $((a,u),(b,v)) \in [\int_A U] \times [\int_B V]$
- $(a,u) \in \int_A U$ and $(b,v) \in \int_B V$
- $a \in A$ and $u \in U_a$ and $b \in B$ and $v \in V_b$
- $(a,b) \in A \times B$ and $(u,v) \in U_a \times V_b$
- $((a,b),(u,v)) \in \int_{(a,b) \in A \times B} U_a \times V_b$

so that the required formula $\sigma((a,u),(b,v)) = ((a,b),(u,v))$ defines a bijection

$$[\textstyle\int_{a \in A} U_a] \times [\int_{b \in B} V_b] \xrightarrow[\approx]{\quad\sigma\quad} \int_{(a,b) \in A \times B} U_a \times V_b \; .$$

Since the integrals of covariant and contravariant integrals agree in dimension 0, the contravariant case is the same.

Before proceeding to higher dimensions, there is a pattern which will persist that may help the reader understand the construction. Let $A \xrightarrow{U_\Box} 0\text{-}\mathsf{Cat}$ with A in $0\text{-}\mathsf{Cat}$. There are evident embeddings for every $a \in A$, and an evident projection:

$$U_a \xrightarrow{\mathsf{emb}_a} \textstyle\int_A U \xrightarrow{\mathsf{proj}} A\,.$$

This will remain true for the integrals of both covariant and contravariant n-functors, but I will not explore it until section 4.

3.3 Integrating Covariant n-Functors

Let $n > 0$ and assume the induction hypothesis in dimension $n-1$.

Let $A \xrightarrow{U_\Box} n\text{-}\mathsf{Cat}$ be an n-functor with A in $n\text{-}\mathsf{Cat}$. The n-category $\int_A U$ is constructed as follows.

The objects are

$$|\textstyle\int_A U| = \int_{|A|} |\Box| U = \int_{a \in |A|} |U_a|,$$

and for objects (a,u) and (b,v), the hom-$(n-1)$-categories are

$$[\textstyle\int_A U][(a,u),(b,v)] = \int^{f \in A(a,b)} U_b(U_f u, v).$$

The identity at an object (a,u) is

$$1_{(a,u)} = (1_a, 1_u),$$

and for objects (a,u), (b,v) and (c,w), the composition $(n-1)$-functor $[\int_A U]^{[(b,v)]}_{(a,u),(c,w)}$ from (a,u) to (c,w) by way of (b,v) is

$$\begin{array}{c}
[\int_A U][(b,v),(c,w)] \times [\int_A U][(a,u),(b,v)] \\
\Big\downarrow{\scriptstyle = \text{ (def. of homs)}} \\
[\int^{g\in A(b,c)} U_c(U_g v, w)] \times [\int^{f\in A(a,b)} U_b(U_f u, v)] \\
\Big\downarrow{\scriptstyle \sigma} \\
\int^{(g,f)\in A(b,c)\times A(a,b)} U_c(U_g v, w) \times U_b(U_f u, v) \\
\Big\downarrow{\scriptstyle \int^{h=gf} I_{g,f}} \\
\int^{h\in A(a,c)} U_c(U_h u, v) \\
\Big\downarrow{\scriptstyle = \text{ (def. of homs)}} \\
[\int_A U][(a,u),(c,w)]
\end{array}$$

where $I_{g,f} = [U_c]^{[U_g v]}_{U_{gf}u,w}(U_c(U_g v, w) \times [U_g]_{U_f u,v})$. Here $[U_c]^{[U_g v]}_{U_{gf}u,w}$ is composition in U_c from $U_{gf}u$ to w by way of $U_g v$ and $[U_g]_{U_f u,v}$ is the action of the n-functor $U_b \xrightarrow{U_g} U_c$ on the hom-$(n-1)$-category $U_b(U_f u, v)$. So, on cells (y,x) of $U_c(U_g v, w) \times U_b(U_f u, v)$ we have

$$\begin{aligned}
I_{g,f}(y,x) &= [U_c]^{[U_g v]}_{U_{gf}u,w}(U_c(U_g v, w) \times [U_g]_{U_f u,v})(y,x) \\
&= [U_c]^{[U_g v]}_{U_{gf}u,w}(y, U_g x) \\
&= y(U_g x).
\end{aligned}$$

If this makes sense, chasing an arbitrary k-cell $((g,q),(f,p))$ from top to bottom, using the inductive formula for mapping contravariant integrals, the composition satisfies the required formula

$$(g,q)(f,p) = (gf, q(U_{\mathsf{dom}_1 g}\, p)),$$

since $\mathsf{dom}_1\, g$ in A is $\mathsf{dom}_0\, g$ in $A(b,c)$.

Now let's check that these definitions make sense. The object-set makes sense by the case $n = 0$, and the integrand $\langle U_b(U_f u, v)\rangle^{f\in A(a,b)}$ used for defining the hom-

$(n-1)$-categories is indeed a contravariant $(n-1)$-category-valued $(n-1)$-functor of $A(a,b)$. The identities exist because

$$1_{(a,u)} = (1_a, 1_u) \in |[\textstyle\int_A U][(a,u),(a,u)]| = \int^{f\in|A(a,a)|} |U_a(U_f u, u)|,$$

since $1_a \in |A(a,a)|$ and $1_u \in |U_a(U_{1_a}u,u)| = |U_a(u,u)|$.

Verifying that the compositions are $(n-1)$-functors by induction, after scrutiny to vet the definition, comes down to verifying the $(n-1)$-naturality of the integrand $I_{g,f}$, which makes sense as

$$\begin{tikzcd} U_c(U_g v, w) \times U_b(U_f u, v) \arrow[d, "U_c(U_g v,w)\times[U_g]_{U_f u,v}"] \\ U_c(U_g v, w) \times U_c(U_g f u, U_g v) \arrow[d, "{[U_c]^{[U_g v]}_{U_g f u,w}}"] \\ U_c(U_{gf} u, w) \end{tikzcd}$$

So we must check that, for any $(k+1)$-cells of A,

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "{[k+1]\Downarrow\varphi}"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "{[k+1]\Downarrow\gamma}"] & c \end{tikzcd},$$

that is, k-cells of $A(b,c) \times A(a,b)$, that, in $(n-1)$-Cat,

$$\begin{tikzcd} U_c(U_{g'} v, w) \times U_b(U_{f'} u, v) \arrow[r, "I_{g',f'}"] \arrow[d, "U_c(U_\gamma v,w)\times U_b(U_\varphi u,v)"', "{[k]}"] & U_c(U_{g'f'} u, w) \arrow[d, "{[k]}"', "U_c(U_{\gamma\varphi}u,w)"] \\ U_c(U_g v, w) \times U_b(U_f u, v) \arrow[r, "I_{g,f}"'] & U_c(U_{gf} u, w) \end{tikzcd}$$

Calculating that, for an arbitrary j-cell (y,x) of $U_c(U_{g'}v,w) \times U_b(U_{f'}u,v)$, we have

$$\begin{aligned} [U_c(U_{\gamma\varphi}u,w)]I_{g',f'}(y,x) &= [U_c(U_{\gamma\varphi}u,w)[y(U_{g'}x)] && \text{(action of } I_{g',f'}) \\ &= y(U_{g'}x)(U_{\gamma\varphi}u) && \text{(action of } U_c(\Box,\Box)) \end{aligned}$$

and

$$
\begin{aligned}
&I_{g,f}(U_c(U_\gamma v, w) \times U_b(U_\varphi u, v))(y, x) \\
&= I_{g,f}(y(U_\gamma v), x(U_\varphi u)) && \text{(actions of } U_c(\Box,\Box),\ U_b(\Box,\Box)) \\
&= y(U_\gamma v)(U_g(x(U_\varphi u))) && \text{(action of } I_{g,f}) \\
&= y(U_\gamma v)(U_g x)(U_g U_\varphi u) && (U_g \text{ is an } n\text{-functor}) \\
&= y(U_\gamma v)(U_g x)(U_{g\varphi} u) && (U \text{ is an } n\text{-functor})
\end{aligned}
$$

this is just that in U_c,

$$
\begin{tikzcd}
U_{gf}u \arrow[r, "U_{g\varphi}u", "{[k]}"'] \arrow[dr, "{[k]}", "U_{\gamma\varphi}u"'] & U_{gf'}u \arrow[r, "U_g x", "{[j]}"'] \arrow[d, "{[k]}"', "U_{\gamma f'}u"] & U_g v \arrow[d, "{[k]}"', "U_\gamma v"] & \\
& U_{g'f'}u \arrow[r, "{[j]}", "U_{g'}x"'] & U_{g'}v \arrow[r, "{[j]}", "y"'] & w
\end{tikzcd}
$$

which follows as the triangle commutes because $\gamma\varphi = \gamma f' \cdot g\varphi$ and U is an n-functor, while the square is the n-naturality of U_γ at x. Thus, the compositions are $(n-1)$-functors.

To conclude that $\int_A U$ is in fact an n-category, we have left only to check the associative and identity laws for composition along 0-cells. On k-cells, $k > 1$ of $\int_A U$

$$
(a, u) \xrightarrow[{[k]}]{(f,p)} (b, v) \xrightarrow[{[k]}]{(g,q)} (c, w) \xrightarrow[{[k]}]{(h,r)} (d, x)
$$

we find that

$$
\begin{aligned}
&(h,r)[(g,q)(f,p)] \\
&= (h,r)(gf, q(U_{\mathsf{dom}_1 g}p)) && \text{(def. of composition)} \\
&= (hgf, r(U_{\mathsf{dom}_1 h}(q(U_{\mathsf{dom}_1 g}p)))) && \text{(def. of composition)} \\
&= (hgf, r(U_{\mathsf{dom}_1 h}q)(U_{\mathsf{dom}_1 h}U_{\mathsf{dom}_1 g}p)) && (U_{\mathsf{dom}_1 h} \text{ is an } n\text{-functor}) \\
&= (hgf, r(U_{\mathsf{dom}_1 h}q)(U_{(\mathsf{dom}_1 h)(\mathsf{dom}_1 g)}p)) && (U \text{ is an } n\text{-functor}) \\
&= (hgf, r(U_{\mathsf{dom}_1 h}q)(U_{\mathsf{dom}_1(hg)}p)) && ((\mathsf{dom}_1 h)(\mathsf{dom}_1 g) = \mathsf{dom}_1(hg)) \\
&= (hg, r(U_{\mathsf{dom}_1 h}q))(f,p) && \text{(def. of composition)} \\
&= [(h,r),(g,q)](f,p) && \text{(def. of composition)}
\end{aligned}
$$

as well as that

$$
\begin{aligned}
1_{(c,w)}(g,q) &= (1_c, 1_w)(g,q) && \text{(def. of } 1_{(c,w)}) \\
&= (1_c g, 1_w(U_{\mathsf{dom}_1 1_c}q)) && \text{(def. of composition)} \\
&= (g, U_{1_c}q) && (\mathsf{dim}\, 1_c = 1) \\
&= (g,q) && (\text{action of } U_{1_c} = 1_{U_c})
\end{aligned}
$$

and that

$$
\begin{aligned}
(g,q)1_{(b,v)} &= (g,q)(1_b, 1_v) && \text{(def. of } 1_{(b,v)}) \\
&= (g1_b, q(U_{dom_1 g}1_v)) && \text{(def. of composition)} \\
&= (g, q1_{U_{dom_1 g}v}) && (U_{\mathsf{dom}_1 g} \text{ is an } n\text{-functor}) \\
&= (g,q) && \text{(identity law)}
\end{aligned}
$$

which completes the proof that $\int_A U$ is an n-category.

We can now state the 1-dimensional covariant integral in full detail!

Details 3.3.1 (1-dimensional covariant integral)**.**

If $n = 1$ then $\int_A U$ has object set

$$|\int_A U| = \{(a,u)|a \in |A|, u \in |U_a|\}$$

and for objects (a,u) and (b,v), hom-sets

$$[\int_A U][(a,u),(b,v)] = \{(f,p)|a \xrightarrow{f} b \text{ in } A, U_f u \xrightarrow{p} v \text{ in } U_b\}.$$

The identity at an object (a,u) is $1_{(a,u)} = (1_a, 1_u)$ and the composition of

$$(a,u) \xrightarrow{(f,p)} (b,v) \xrightarrow{(g,q)} (c,w)$$

is $(g,q)(f,p) = (gf, q(U_g p))$; the picture in U_c is

$$U_{gf}u \xrightarrow{U_g p} U_g v \xrightarrow{q} w\ .$$

The idea is that to map u in U_a to v in U_b we first push u into U_b via some functor U_f, and then pick an arrow p in U_b from $U_f u$ to v.

Example 3.3.2 (Subspaces)**.**

Let $2^{\Box}$ be the covariant power-set functor $\mathsf{Top} \longrightarrow \mathsf{Ord} \leq \mathsf{Cat}$, where $\mathsf{Ord} \leq \mathsf{Cat}$ is the subcategory of (pre)orders (categories in which every hom-set is either a singleton or empty). Thus, for any space X with subsets A and B there is an arrow from A to B in 2^X precisely when $A \subseteq B$, and for any map $X \xrightarrow{f} Y$ in Top,

$$2^f A = \langle y \in Y | y = fx \text{ for some } x \in A \rangle$$

is the direct image of A.

The category $\int_{X \in \mathsf{Top}} 2^X$ is the category of spaces and subspaces. Indeed,

$$|\int_{X\in\mathsf{Top}} 2^X| = \langle (X,A) | X \in |\,\mathsf{Top}\,|, A \in |2^X| \rangle,$$

and in this category, $(X,A) \xrightarrow{(f,i)} (Y,B)$ means that $X \xrightarrow{f} Y$ in Top, that $A \in |2^X|$, and that $2^f A \xrightarrow{i} B$ in 2^Y, i.e. $2^f A \subseteq B$.

Example 3.3.3 (Covering Spaces)**.**

Let $E \xrightarrow{\pi} B$ be a covering space, and define a functor $\Pi_1 X \xrightarrow{E_\Box} \mathsf{Set} \leq \mathsf{Cat}$ as follows. For $x \in |\Pi_1 X| = X$, let $E_x = \pi^{-1}\langle x \rangle$. For $x \xrightarrow{\gamma} y$ in $\Pi_1 X$, define the function $E_x \xrightarrow{E_\gamma} E_y$ on $p \in E_x$ by $E_\gamma p = \mathsf{cod}[\gamma, p]$, where $[\gamma, p]$ denotes the unique lift of γ with domain p in $\Pi_1 E$. This makes sense because $\pi\,\mathsf{cod}[\gamma,p] = \mathsf{cod}\,\pi[\gamma,p] = \mathsf{cod}\,\gamma = y$.

To see that $E_\Box$ is a functor, let $x \xrightarrow{\gamma} y \xrightarrow{\delta} z$ in $\Pi_1 B$. For $p \in E_x$ we have

$$\begin{aligned}
E_{\delta\gamma} p &= \mathsf{cod}[\delta\gamma, p] && \text{(def. of } E\text{)}\\
&= \mathsf{cod}([\delta, \mathsf{cod}[\gamma,p]][\gamma,p]) && \text{(uniqueness of lifts)}\\
&= \mathsf{cod}[\delta, \mathsf{cod}[\gamma,p]] && (\mathsf{cod}\text{ of composite})\\
&= E_\delta\,\mathsf{cod}[\gamma,p] && \text{(def. of } E\text{)}\\
&= E_\delta E_\gamma p && \text{(def. of } E\text{)}
\end{aligned}$$

and also $E_{1_x} p = \mathsf{cod}[1_x, p] = \mathsf{cod}\,1_p = p = 1_{E_x} p$.

Integrating, we have $\mathcal{I} = \int_{x\in\Pi_1 B} E_x$, which turns out to be (isomorphic to) $\Pi_1 E$. Indeed, a cell $(x,p) \xrightarrow{(\gamma,\alpha)} (y,q)$ of $\mathcal{I}$ means that $x \xrightarrow{\gamma} y$ in $\Pi_1 B$, $\pi p = x$, $\pi q = y$ and in (the set) E_y we have $E_\gamma p = \mathsf{cod}[\gamma,p] \xrightarrow[=]{\alpha} q$.

Mutually inverse functors $\mathcal{I} \xrightarrow{\Theta} \Pi_1 E \xrightarrow{\Phi} \mathcal{I}$ are given by $\Theta(x,p) = p$, $\Theta(\gamma, 1_q) = [\gamma, p]$ (where $(x,p) = \mathsf{dom}(\gamma,\alpha) = \mathsf{dom}(\gamma, 1_q)$), $\Phi p = (\pi p, p)$, and $\Phi\alpha = (\pi\alpha, 1_{\mathsf{cod}\,\alpha})$. The pleasure of checking these details is left to the reader.

Example 3.3.4 (Slice/Comma Categories).

For any functor $A \xrightarrow{N} B$ and object b of B, we can form the "slice" or "comma" category $\int_{a \in A} B(b, Na)$. The objects are pairs (g, i) with $g \in |A|$ and $b \xrightarrow{i} Ng$ in B. An arrow $(g, i) \xrightarrow{(f,z)} (h, j)$ consists of an arrow $g \xrightarrow{f} h$ in A and an arrow

$$B(b, Nf)i = (Nf)i \xrightarrow{z} j \text{ in } B(b, Nh).$$

But $B(b, Nh)$ is a 0-category regarded as a 1-category, so the only arrows are identities: $z = 1_j$ and $j = (Nf)i$. The only arrows take the form

$$(g, i) \xrightarrow{(f, 1_{(Nf)i})} (h, (Nf)i),$$

indicating that, in B,

$$\begin{tikzcd} & Ng \arrow[dd, "Nf"] \\ b \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & Nh \end{tikzcd}$$

Composition is by "stacking" these triangles. Note: even if $Nf = Nf'$, the arrows $(f, 1_{(Nf)i})$ and $(f', 1_{(Nf')i})$ are distinct unless $f = f'$.

Example 3.3.5 (Free Groups).

In the previous example, take $A = \mathsf{Grp}$, the category of groups, $B = \mathsf{Set}$ and let N be the forgetful functor. Let b be any set and $b \xrightarrow{i} NF_b$ be the usual inclusion of generators into the group F_b freely generated by b. The universal property of i may be expressed by observing that (F_b, i) is initial in $\int_{G \in \mathsf{Grp}} \mathsf{Set}(b, NG)$. Since any two initial objects are isomorphic in a unique way, we see that any two groups freely generated by b are isomorphic in a unique way commuting with the insertions of generators. (This is typical of the use of slice categories.)

Example 3.3.6 (Variance Categories).

Let G be a group (or monoid) acting on a category C. This is to say, there is a functor $G \xrightarrow{U_\square} \mathsf{Cat}$ with $U_\star = C$ for the unique object $\star$ of G, regarded as a category. Call the elements (arrows) of G variances on C. I will call $\int_G U$ the G-varianced version of C.

For example, in the case of $G = \{\pm 1\}$ acting on the category Cat by $-C = \overline{C}$ and $-F = \overline{F}$ for any functors $D \xrightarrow{F} C$ in Cat. The varianced version of Cat obtained has one object $(\star, C)$ for every $C \in |\,\mathsf{Cat}\,|$, and the arrows are of two types, either of the type $(\star, D) \xrightarrow{(+1,F)} (\star, C)$, meaning that $D \xrightarrow{F} C$ in Cat (F is covariant) or of the type $(\star, D) \xrightarrow{(-1),F)} (\star, C)$, meaning that $\overline{D} \xrightarrow{F} C$ in Cat (F is contravariant). The composition in the G-varianced version of Cat is the old-fashioned calculus of mixed-variance functors, e.g. composing contravariant with contravariant gives covariant, etc. as has been encoded by the arithmetic of G.

A similar example is to let $G = \{\pm 1\}$ act on the category Vct of complex vector spaces by assigning to each vector-space V its conjugate $-V$ with the same addition, but scalar product given by

$$(\alpha v \text{ in } -V) = (\overline{\alpha} v \text{ in } V)$$

for scalars α and vectors v, and by assigning to each linear mapping $V \xrightarrow{T} W$ the same map, but with domain $-V$ and codomain $-W$. The resulting G-varianced category again has one object per vector space, and one type of arrow per element of G. The arrows coresponding to $+1$ are the linear maps, and the arrows coresponding to -1 are the antilinear maps. The arithmetic of G again encodes the variance of the composition of two such maps.

It may feel quite different, but suppose C is actually a group (or monoid). Then U is an action on the left by G on C, and the resulting G-varianced category is the

left-handed version of the semidirect product group $C \rtimes_U G$.

Example 3.3.7 (Idempotents in Semigroups)**.**

There is a functor $\mathsf{Sgp} \xrightarrow{E} \mathsf{Cat}$ assigning to every semigroup S the category ES of idempotents in S. The objects of ES are the idempotent elements of S, and an arrow $e \xrightarrow{a} f$ in ES is an element $a \in S$ with $fae = a$. The composition in ES of $e \xrightarrow{a} f \xrightarrow{b} g$ is $e \xrightarrow{ba} g$, and the identities are $e \xrightarrow{1_e = e} e$. This captures order information about the idempotents of S, e.g. $e \xrightarrow{f} e$ in ES means that $f \leq e$, while $e \xrightarrow{f} f$ in ES means that $fe = f$.

Integrating yields a category of all idempotents in all semigroups. In $\int_{\mathsf{Sgp}} E$ the arrows $(S, e) \xrightarrow{(\varphi, a)} (T, f)$, from an idempotent e of S to an idempotent f of T, consist of a morphism $S \xrightarrow{\varphi} T$ in Sgp and an element $a \in T$ such that $\varphi e \xrightarrow{a} f$ in ET, i.e. $fa(\varphi e) = a$. This relates to Nambooripad's work in [10].

3.4 Integrating Contravariant n-Functors

Let $n > 0$ and assume the induction hypothesis in dimension $n - 1$, and that we can integrate covariant n-functors, which is the first half of step 1 at n.

Let $\overline{B} \xrightarrow{V_\Box} n\text{-}\mathsf{Cat}$ be an n-functor with B in $n\text{-}\mathsf{Cat}$. Form the n-functor

$$\overline{B}^* \xrightarrow{V^*} n\text{-}\mathsf{Cat}^* \xrightarrow{\overline{\Box}^*} n\text{-}\mathsf{Cat} ,$$

and integrate it to define

$$\int^B V = \overline{\int_{\overline{B}^*} \overline{\Box}^* V^*}^* = \overline{\int_{b \in \overline{B}^*} \overline{V_b}^*}^* .$$

Any k-cells of $\int^B V$, composable along a 0-cell,

$$(a, u) \xrightarrow[{[k]}]{(f,p)} (b, v) \xrightarrow[{[k]}]{(g,q)} (c, w) ,$$

are k-cells of $\int_{b\in\overline{B}^*}\overline{V_b}^*$,

$$(c,w)\xrightarrow[{[k]}]{(g,q)}(b,v)\xrightarrow[{[k]}]{(f,p)}(a,u)\ ,$$

with composition along (b,v) in $\int_{b\in\overline{B}^*}\overline{V_b}^*$, given by

$$(f,p)(g,q)=(fg,p(\overline{V_{\mathsf{dom}_1 f}}^*q)),$$

where $\mathsf{dom}_1 f$ and fg are computed in $\overline{B}^*$ and $p(\overline{V_{\mathsf{dom}_1 f}}^*q)$ is computed in $\overline{V_a}^*$. Thus, in $\int^B V=\overline{\int_{\overline{B}^*}\overline{V_b}^*}^*$,

$$(g,q)(f,p)=(gf,(V_{\mathsf{cod}_1 f}q)p),$$

computing gf and $\mathsf{cod}_1 f$ in B instead, and $(V_{\mathsf{cod}_1 f}q)p$ in V_a instead. This is the formula the induction requires.

So I fibbed when I said there were two different integrals, but it was a white lie – attempting this induction with only one integral in hand would result in a sort of duality-blindness. The slight complication of a second integral seems more than off-set by organizing these last details away from the main inductive loop.

Moreover, contravariant integrals seem common in practice, so it seems worth documenting their properties instead of deriving them each time.

Note that

$$\begin{aligned}\left|\int^B V\right| &= \left|\overline{\int_{b\in\overline{B}^*}\overline{V_b}^*}^*\right|\\ &= \left|\int_{b\in\overline{B}^*}\overline{V_b}^*\right|\\ &= \int_{b\in|\overline{B}^*|}|\overline{V_b}^*|\\ &= \int_{b\in|B|}|V_b|\\ &= \int^{b\in|B|}|V_b|\end{aligned}$$

as in the covariant case. There is also a useful formula for the hom-$(n-1)$-categories of $\int^B V$. Let (a,u) and (b,v) be objects. Now,

$$\begin{aligned}
\left[\int^B V\right][(a,u),(b,v)] &= \overline{\int_{b\in\overline{B}^*} \overline{V_b}^*}^* [(a,u),(b,v)] \\
&= \overline{\left[\int_{b\in\overline{B}^*} \overline{V_b}^*\right][(b,v),(a,u)]}^* \\
&= \overline{\int^{f\in\overline{B}^*(b,a)} \overline{V_a}^*[\overline{V_f}^* v,u]}^* \\
&= \overline{\int^{f\in\overline{B(a,b)}^*} \overline{V_a[u,V_f v]}^*}^* \\
&= \int_{f\in B(a,b)} V_a[u,V_f v]
\end{aligned}$$

Again we are in position to detail the 1-dimensional case for clarity.

Details 3.4.1 (1-Dimensional Contravariant Integral Categories)**.**

Let $n=1$. Thus $\int^B V = \overline{\int_{b\in\overline{B}} \overline{V_b}}$, since there are no cells above dimension 1 to reverse. We find the object set to be

$$|\textstyle\int^B V| = \{(a,u)|a\in|B| \text{ and } u\in|V_a|\}$$

and the hom-set from an object (a,u) to another (b,v) to be

$$[\textstyle\int^B V][(a,u),(b,v)] = \{(f,p)|a\xrightarrow{f}b \text{ in } B \text{ and } u\xrightarrow{p}V_f b \text{ in } V_a\}$$

with the composition of

$$(a,u)\xrightarrow{(f,p)}(b,v)\xrightarrow{(g,q)}(c,w)$$

given by $(g,q)(f,p)=(gf,(V_f q)p)$. The picture in V_a is

$$u\xrightarrow{p}V_f v\xrightarrow{V_f q}V_f V_g w = V_{gf} w\ .$$

The idea is that to map u in V_a to v in V_b we first pull v back into V_a via some functor V_f, and then we pick an arrow p.

Example 3.4.2 (B-Graphs)**.**

For any base category B, the category of B-graphs is

$$B\,\text{-}\mathsf{Grph} = \int^{V \in \mathsf{Set}} \mathsf{Cat}(V \times V, B).$$

Indeed, the 0-cells are $(V, F_\Box)$ where V is a set and $F \in |\,\mathsf{Cat}(V \times V, B)|$ is a family $F = \langle F_{a,b} \rangle_{(a,b) \in V \times V}$ of objects of B, which we might as well write as $(V, F) = (|\Gamma|, \langle \Gamma(a,b) \rangle_{a,b \in |\Gamma|})$, as in the introduction.

The 1-cells $\Gamma \xrightarrow{\Phi} \Delta$ are pairs $\Phi = (f, t)$ with $|\Gamma| \xrightarrow{f} |\Delta|$ in Set and t an element of

$$|\,\mathsf{Cat}(|\Gamma| \times |\Gamma|, B)[\langle \Gamma(a,b) \rangle_{(a,b) \in |\Gamma| \times |\Gamma|}, [\mathsf{Cat}(f \times f, B)] \langle \Delta(c,d) \rangle_{(c,d) \in |\Delta| \times |\Delta|}]|$$

which is just

$$|\,\mathsf{Cat}(|\Gamma| \times |\Gamma|, B)[\langle \Gamma(a,b) \rangle_{(a,b) \in |\Gamma| \times |\Gamma|}, \langle \Delta(fa, fb) \rangle_{(a,b) \in |\Gamma| \times |\Gamma|}].$$

Because $|\Gamma| \times |\Gamma|$ is 0-dimensional, the naturality condition for t is vacuous, so we might as well write $\Phi = (f,t) = (|\Phi|, \langle \Gamma(a,b) \xrightarrow[a,b]{\Phi} \Delta(|\Phi|a, |\Phi|b) \rangle_{a,b \in |\Gamma|})$, as in the introduction.

We now see some details slighted in the introduction. Both

$$1_\Gamma = (1_{|\Gamma|}, \langle 1_{\Gamma(a,b)} \rangle_{a,b \in |\Gamma|}), \text{ i.e. } |1_\Gamma| = 1_{|\Gamma|} \text{ and } (1_\Gamma)_{a,b} = 1_{\Gamma(a,b)},$$

and for composable arrows in $B\,\text{-}\mathsf{Grph}$,

$$\Gamma \xrightarrow{\Phi} \Delta \xrightarrow{\Theta} \Upsilon\ ,$$

we have $\Theta\Phi = (|\Theta||\Phi|, \langle \Theta_{\Phi a, \Phi b} \Phi_{a,b} \rangle_{a,b \in |\Gamma|})$, i.e. $|\Theta\Phi| = |\Theta||\Phi|$ and $[\Theta\Phi]_{a,b} = \Theta_{\Phi a, \Phi b} \Phi_{a,b}$, the picture in B being

$$\Gamma(a,b) \xrightarrow{\Phi_{a,b}} \Delta(\Phi a, \Phi b) \xrightarrow{\Theta_{\Phi a,\Phi b}} \Upsilon(\Theta\Phi a, \Theta\Phi b)\ .$$

A direct calculation of the associative and identity laws would be fussy, but the laws follow directly from the construction of the contravariant integral.

Example 3.4.3 (Rings as Modules)**.**

The contravariant functor $\overline{\mathsf{Rng}} \xrightarrow{\square\text{-}\mathsf{Mod}} \mathsf{Cat}$ assigning each ring R its category $R\text{-}\mathsf{Mod}$ of R-modules is well-known. (Actually, several are known, depending on whether we ask rings to be commutative, associative or unital, for example, but in this case we don't need to know.) We may thus form $\mathcal{M} = \int^{R \in \mathsf{Rng}} R\text{-}\mathsf{Mod}$, the category of all modules over all rings.

This is usually the wrong thing to do, since ring-theorists typically use $\square\text{-}\mathsf{Mod}$ as an invariant of rings, studying $R\text{-}\mathsf{Mod}$ to study R.

But for considering R as an R-module, it's just right. Define a functor $\mathsf{Rng} \xrightarrow{\Delta} \mathcal{M}$ by mapping $R \xrightarrow{\varphi} S$ in Rng to

$$\Delta R = (R,R) \xrightarrow{\Delta\varphi = (\varphi,\varphi)} \Delta S = (S,S)$$

which is an arrow of $\mathcal{M}$ because φ is a ring homomorphism from R to S and $R \xrightarrow{\varphi} (f\text{-}\mathsf{Mod})S$ is R-linear. Indeed, for $r, s \in R$,

$$\varphi(rs) = ((\varphi r)(\varphi s) \text{ in } S) = (r(\varphi s) \text{ in } (\varphi S)\text{-}\mathsf{Mod})$$

because φ is a ring homomorphism and by the definition of the scalar product of $(\varphi\text{-}\mathsf{Mod})S$.

Similar remarks hold for the case of the action by left translation of a group, monoid or semigroup on itself.

Example 3.4.4 (Ringed Spaces)**.**

The contravariant functor assigning each topological space X its category Sh_X of sheaves of rings is well-known (to some algebraic geometers). The category of ringed spaces, well known to the same people, is just $\int^{\mathsf{Top}} \mathsf{Sh}_\square$. This and further applications in algebraic geometry can be found in [11].

3.5 Mapping Integrals

Assume $n > 0$, the induction hypothesese in dimension $n-1$ and step 1 in dimension n. Let an n-natural transformation

$$\begin{tikzcd} A \arrow[r, "U_\square"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow P_\square"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ B \arrow[r, "V_\square"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

be given with $A \xrightarrow{F} B$ in n-Cat.

Define the n-functor $\int_A U \xrightarrow{\int_F P} \int_B V$ on objects by $|\int_F P| = \int_{|F|} |\square| P$, i.e. as

$$|\int_A U| = \int_{a\in|A|} |U_a| \xrightarrow{\int_{b=|F|a} |P_a|} |\int_{b\in|B|} |V_b| = |\int_B V|,$$

giving the desired action on 0-cells by the 0-dimensional case: $|\int_F P|(a,u) = (Fa, P_a u)$. On the hom-$(n-1)$-category from (a,u) to (b,v), inductively define

$$\begin{tikzcd} [\int_A U][(a,u),(b,v)] \arrow[r, "{[\int_F P]_{(a,u),(b,v)}}"] \arrow[d, "="'] & [\int_B V][(Fa,P_a u),(Fb,P_b v)] \arrow[d, "="] \\ \int^{f\in A(a,b)} U_b(U_f u, v) \arrow[r, "{\int^{g=F_{a,b}f} [P_b]_{U_f u,v}}"'] & \int^{g\in B(Fa,Fb)} V_{Fb}(V_{Ff} P_a u, P_b v) \end{tikzcd}$$

Now, if this makes sense, chasing a cell (f,p) of $[\int_A U][(a,u),(b,v)]$ we find that

$$[\int_F P](f,p) = (Ff, P_b p) = (Ff, P_{cod_0 f} p),$$

which is the desired formula on cells.

Since $A(a,b) \xrightarrow{F_{a,b}} B(Fa,Fb)$ is an $(n-1)$-functor, it remains only to check that $U_b(U_f u, v) \xrightarrow{[P_b]_{U_f u,v}} V_{Fb}(V_{Ff}P_a u, P_b v)$ is $(n-1)$-natural in f, i.e. that for any $(k+1)$-cell of A,

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "{[k{+}1]\Downarrow\varphi}"] & b \end{tikzcd},$$

that is, for any k-cell φ of $A(a,b)$, that in $(n-1)$-Cat,

$$\begin{tikzcd} U_b(U_{f'}u,v) \arrow[r, "{[P_b]_{U_{f'}u,v}}"] \arrow[d, "{U_b(U_\varphi u,v)}"', "{[k]}"] & V_{Fb}(V_{Ff'}P_a u, P_b v) \arrow[d, "{[k]}"', "{V_{Fb}(V_{F\varphi}P_a u, P_b v)}"] \\ U_b(U_f u,v) \arrow[r, "{[P_b]_{U_f u,v}}"'] & V_{Fb}(V_{Ff}P_a u, P_b v) \end{tikzcd}$$

which makes sense because

$$U_b(U_f u, v) \xrightarrow{[P_b]_{U_f u,v}} V_{Fb}(P_b U_f u, P_b v) = V_{Fb}(V_{Ff}P_a u, P_b v)$$

(and likewise for f') by the naturality of P at f (or f').

Accordingly, let x by any cell of $U_b(U_{f'}u, v)$ and calculate that

$$\begin{aligned}
[P_b]_{U_f u,v}[U_b(U_\varphi u, v)]x &= [P_b]_{U_f u,v}[x(U_\varphi u)] && \text{(action of } U_b(\Box,\Box))\\
&= P_b[x(U_\varphi u)] && ([P_b]_{U_f u,v} \text{ is } P_b \text{ on a hom})\\
&= [P_b x][P_b U_\varphi u] && (P_b \text{ is an } n\text{-functor})\\
&= [P_b x][V_{F\varphi}P_a u] && (n\text{-naturality of } P \text{ at } \varphi)\\
&= [V_{Fb}(V_{F\varphi}P_a u, P_b v)][P_b x] && \text{(action of } V_{Fb}(\Box,\Box))\\
&= [V_{Fb}(V_{F\varphi}P_a u, P_b v)][P_b]_{U_f u,v}x && ([P_b]_{U_f u,v} \text{ is } P_b \text{ on a hom})
\end{aligned}$$

So at least $\int_F P$ is a $(n-1)$-Cat-graph morphism – it has hom-$(n-1)$-functors. It remains only to see that it preserves the composition along 0-cells and identities at 0-cells. Accordingly, let k-cells of $\int_A U$,

$$(a,u)\xrightarrow[{[k]}]{(f,p)}(b,v)\xrightarrow[{[k]}]{(g,q)}(c,w),$$

be given, and calculate that

$$\begin{aligned}
\left[\int_F P\right][(g,q)(f,p)] &= \left[\int_F P\right](gf, q(U_{\mathsf{dom}_1 g}p)) && \text{(def. of composition)}\\
&= (F(gf), P_c(q(U_{\mathsf{dom}_1 g}p))) && \text{(action of } \textstyle\int_F P)\\
&= ((Fg)(Ff), (P_c q)(P_c U_{\mathsf{dom}_1 g}p)) && (F \text{ and } P_c \text{ are } n\text{-functors})\\
&= ((Fg)(Ff), (P_c q)(V_{F\,\mathsf{dom}_1 g}P_b p)) && (n\text{-naturality of } P \text{ at } \mathsf{dom}_1 g)\\
&= ((Fg)(Ff), (P_c q)(V_{\mathsf{dom}_1 Fg}P_b p)) && (F\,\mathsf{dom}_1 g = \mathsf{dom}_1 Fg)\\
&= (Fg, P_c q)(Ff, P_b p) && \text{(def. of composition)}\\
&= \left(\left[\int_F P\right](g,q)\right)\left(\left[\int_F P\right](f,p)\right) && \text{(action of } \textstyle\int_F P)
\end{aligned}$$

and that

$$\begin{aligned}
[\int_F P]1_{(b,v)} &= [\int_F P](1_b, 1_v) && \text{(def. of } 1_{(b,v)})\\
&= (F1_b, P_b 1_v) && \text{(action of } \textstyle\int_F P)\\
&= (1_{Fb}, 1_{P_b v}) && (F \text{ and } P_b \text{ are } n\text{-functors})\\
&= 1_{(Fb, P_b v)} && \text{(def. of } 1_{(Fb,P_b v)})\\
&= 1_{[\int_F P](b,v)} && \text{(action of } \textstyle\int_F P)
\end{aligned}$$

Therefore, $\int_F P$ is indeed an n-functor.

For the contravariant case, let an n-natural transformation

$$
\begin{tikzcd}
\overline{C} \arrow[r, "U_\Box"] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\Downarrow Q_\Box"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
\overline{D} \arrow[r, "V_\Box"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}
$$

be given with $C \xrightarrow{G} D$ in n-Cat.

Applying $\Box^*$ and composing with $\overline{\Box}^*$ we have an n-natural transformation

$$
\begin{tikzcd}
\overline{C}^* \arrow[r, "U^*"] \arrow[d, "\overline{G}^*"'] \arrow[dr, phantom, "\Downarrow Q^*"] & n\text{-}\mathsf{Cat}^* \arrow[r, "\overline{\Box}^*"] \arrow[d, "="] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
\overline{D}^* \arrow[r, "V^*"'] & n\text{-}\mathsf{Cat}^* \arrow[r, "\overline{\Box}^*"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}
$$

vetting the definition $\int^G Q = \overline{\int_{\overline{G}^*} \overline{\Box}^* Q^*}^*$, which we may also write as

$$\int^C U = \overline{\int_{c\in\overline{C}^*} \overline{U_c}^*}^* \xrightarrow[\overline{\int_{d=\overline{G}^* c} \overline{Q_c}^*}^*]{} \overline{\int_{d\in\overline{D}^*} \overline{V_d}^*}^* = \int^D V\,.$$

It satisfies the required formula on k-cells (g, q), $k > 0$, because $(\mathsf{cod}_0\, g$ in $\overline{C}^*) =$ $(\mathsf{dom}_0\, g$ in $C)$.

Example 3.5.1 (Right-Handed Semidirect Products)**.**

Let M be a monoid acting on the *right* by endomorphisms on another monoid S. This amounts to a functor $\overline{M} \xrightarrow{U_\Box} \mathsf{Mon}$ to the subcategory Mon of monoids within Cat, mapping the unique object $\star_M$ of M to S. The (right-handed) semidirect product is $M \rtimes_U S = \int^M U$. (Note to group theorists: the left-handed semidirect product of monoids is different, even though for groups it is the same. The left-handed semidirect product is a covariant integral, from the previous example 3.3.6.) For the record, for all $(m, s), (m', s') \in M \rtimes_U S$ their product is

$$(m, s)(m', s') = (mm', (U_{m'} s)s') = (mm', s^{m'} s'),$$

writing the action exponentially.

Let N be another monoid, and $\overline{N} \xrightarrow{V_\Box} \mathsf{Mon}$ be another such action, this time on a monoid $T = V_{\star_N}$ where $\star_N$ denotes the unique object of N. Given a natural transformation

$$\begin{tikzcd} \overline{M} \arrow[r, "U_\Box"] \arrow[d, "\overline{\varphi}"'] \arrow[dr, phantom, "\Downarrow\pi_\Box"] & \mathsf{Mon} \arrow[d, "="] \\ \overline{N} \arrow[r, "V_\Box"'] & \mathsf{Mon} \end{tikzcd}$$

with φ a morphism of monoids, we obtain a monoid morphism

$$M \rtimes_U S = \int^M U \xrightarrow{\varphi \rtimes \pi = \int^\varphi \pi} \int^N V = N \rtimes_V T \,.$$

Since M has only one object $\star_M$, π is completely determined by the morphism

$$S = U_{\star_M} \xrightarrow{\pi_{\star_M}} V_{\varphi\star_M} = V_{\star_N} = T\,,$$

to which I will now refer, less pedantically as simply π. The naturality of π states that, for $m \in M$ (that is, $\star_M \xrightarrow{m} \star_M$), we have $\pi U_m = V_{\varphi m}\pi$. Writing the actions as exponents, this says that for all $m \in M$ and $s \in S$, $\pi(s^m) = (\pi s)^{\varphi m}$.

Checking directly that $(\varphi \rtimes \pi)(m,s) = (\varphi m, \pi s)$ for $(m,s) \in M \rtimes_U S$ defines a monoid morphism may entertain the reader, but we have already proved this.

3.6 Products of Integrals

Assume $n > 0$, the induction hypothesis in dimension $n-1$ and steps 1 and 2 in dimension n.

Given n-functors $A \xrightarrow{U_\Box} n\text{-}\mathsf{Cat}$ and $B \xrightarrow{V_\Box} n\text{-}\mathsf{Cat}$, with both A and B in $n\text{-}\mathsf{Cat}$, we seek an isomorphism of n-categories

$$[\textstyle\int_{a\in A} U_a] \times [\int_{b\in B} V_b] \xrightarrow[\approx]{\sigma} \int_{(a,b)\in A\times B} U_a \times V_b \,,$$

acting on cells by $\sigma((a,u),(b,v)) = ((a,b),(u,v))$.

We have an n-natural transformation

$$\begin{tikzcd} A \times B \arrow[r, "\langle U_a \times V_b\rangle_{(a,b)\in A\times B}"] \arrow[d, "\mathsf{proj}"'] \arrow[dr, phantom, "\Downarrow\pi_{\Box}"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ A \arrow[r, "U"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

where proj is the projection, by taking $\pi_{a,b}$ as the projection $U_a \times V_b \xrightarrow{\pi_{a,b}} U_a$ for every $(a,b) \in |A \times B|$. Indeed, for any k-cell of $A \times B$,

$$(a,b) \xrightarrow[{[k]}]{(f,g)} (a',b')$$

the very definition of $U_f \times V_g$ is as the unique k-cell so that, in $n\text{-}\mathsf{Cat}$,

$$\begin{tikzcd} U_a \arrow[d, "U_f"', "{[k]}"] & U_a \times V_b \arrow[l, "\pi_{a,b}"'] \arrow[r, "\rho_{a,b}"] \arrow[d, "U_f\times V_g"', "{[k]}"] & V_b \arrow[d, "{[k]}"', "V_g"] \\ U_{a'} & U_{a'} \times V_{b'} \arrow[l, "\pi_{a',b'}"] \arrow[r, "\rho_{a',b'}"'] & V_{b'} \end{tikzcd}$$

where the top and bottom rows are product diagrams. The left-hand square proves π to be n-natural.

Consequently, there is an n-functor

$$\textstyle\int_{(a,b)\in A\times B} U_a \times V_b \xrightarrow{\kappa_0} \int_A U$$

given by $\kappa_0 = \int_{\mathsf{proj}} \pi$. On k-cells $((a,b),(u,v))$,

$$\kappa_0((a,b),(u,v)) = (\mathsf{proj}(a,b), \pi_{\mathsf{cod}_0 a, \mathsf{cod}_0 b}(u,v)) = (a,u).$$

Similarly, we have an n-functor

$$\textstyle\int_{(a,b)\in A\times B} U_a \times V_b \xrightarrow{\kappa_1} \int_B V$$

given by $\kappa_1 = \int_{\mathsf{proj}'} \rho$, where $A \times B \xrightarrow{\mathsf{proj}'} B$ is the other projection from $A \times B$ and ρ is defined above, with π. Again, for all cells $((a,b),(u,v))$ of any dimension,

$$\kappa_1((a,b),(u,v)) = (b,v).$$

There is thus a unique n-functor τ so that, in $n\text{-}\mathsf{Cat}$

$$\begin{tikzcd}
 & \int_{(a,b)\in A\times B} U_a \times V_b \arrow[dl, "\kappa_0"'] \arrow[dr, "\kappa_1"] \arrow[dd, "\tau"] & \\
\int_A U & & \int_B V \\
 & {[\int_A U] \times [\int_B V]} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where the unlabelled arrows are the projections. Chasing an arbitrary cell we find

$$\tau((a,b),(u,v)) = (\kappa_0((a,b),(u,v)), \kappa_1((a,b),(u,v))) = ((a,u),(b,v)).$$

Now all that remains to show is that τ is invertible; then we can put $\sigma = \tau^{-1}$.

Consider the action of τ on an arbitrary hom-$(n-1)$-category

$$\begin{aligned}
&\left[\int_{(a,b)\in A\times B} U_a \times V_b\right][((a,b),(u,v)),((c,d),(w,x))] \\
&= \left[\int_{c\in A\times B} U_{\mathsf{proj}\,c} \times V_{\mathsf{proj}'\,c}\right][((a,b),(u,v)),((c,d),(w,x))] \\
&= \int^{h\in[A\times B][(a,b),(c,d)]} [U_{\mathsf{proj}(c,d)} \times V_{\mathsf{proj}'(c,d)}][(U_{\mathsf{proj}\,h} \times V_{\mathsf{proj}'\,h})(u,v),(w,x)] \\
&= \int^{(f,g)\in[A\times B][(a,b),(c,d)]} [U_c \times V_d][(U_f \times V_g)(u,v),(w,x)] \\
&= \int^{(f,g)\in A(a,c)\times B(b,d)} [U_c \times V_d][(U_f u, V_g v),(w,x)] \\
&= \int^{(f,g)\in A(a,c)\times B(b,d)} U_c(U_f u, w) \times V_d(V_g v, x)
\end{aligned}$$

which $\tau_{((a,b),(u,v)),((c,d),(w,x))}$ maps to

$$\begin{aligned}
&\left[\left[\int_A U\right] \times \left[\int_B V\right]\right] [\tau((a,b),(u,v)), \tau((c,d),(w,x))] \\
&= \left[\left[\int_A U\right] \times \left[\int_B V\right]\right] [((a,u),(b,v)),((c,w),(d,x))] \\
&= \left[\int_A U\right] [(a,u),(c,w)] \times \left[\int_B V\right] [(b,v),(d,x)] \\
&= \left[\int^{f\in A(a,c)} U_c(U_f u, c)\right] \times \left[\int^{g\in B(b,d)} V_d(V_g v, x)\right]
\end{aligned}$$

On hom-$(n-1)$-categories, τ is thus σ^{-1} from the contravariant case in dimension $n-1$, so the induction hypothesis applies to show that every hom-$(n-1)$-functor of τ is invertible. But $|\tau|$ is also invertible, by the dimension 0 case. Therefore τ is invertible, concluding the covariant case.

Changing gears, now suppose we have n-functors $\overline{A} \xrightarrow{U_\square} n\text{-}\mathsf{Cat}$ and $\overline{B} \xrightarrow{V_\square} n\text{-}\mathsf{Cat}$, with both A and B in $n\text{-}\mathsf{Cat}$; we now seek an isomorphism of n-categories

$$[\textstyle\int^{a\in A} U_a] \times [\int^{b\in B} V_b] \xrightarrow[\approx]{\sigma} \int^{(a,b)\in A\times B} U_a \times V_b \,,$$

acting on cells by $\sigma((a,u),(b,v)) = ((a,b),(u,v))$.

Well, it's easy. Observing that $\overline{C \times D}^* = \overline{C}^* \times \overline{D}^*$ for all n-categories C and D, we just define

$$\begin{tikzcd}
{[\int^{a\in A} U_a] \times [\int^{b\in B} V_b]} \arrow[rr, "\sigma", "\approx"'] \arrow[d, "="'] & & {\int^{(a,b)\in A\times B} U_a \times V_b} \arrow[d, "="] \\
\overline{[\int_{a\in\overline{A}^*} \overline{U_a}^*] \times [\int_{b\in\overline{B}^*} \overline{V_b}^*]}^* \arrow[rr, "\approx", "\overline{\sigma}^*"'] & & \overline{\int_{(a,b)\in\overline{A}^*\times\overline{B}^*} \overline{U_a \times V_b}^*}^*
\end{tikzcd}$$

where the lower σ comes from the covariant case.

This concludes the main induction we embarked on back in section 3.1. A celebration is in order. If you were disturbed by the notation's lack of reference to the

dimension, you may prefer to skip ahead section 3.9 next, where I show that the integrals are compatible with the dimensional tower

$$0\text{-}\mathsf{Cat} \leq 1\text{-}\mathsf{Cat} \leq 2\text{-}\mathsf{Cat} \leq \cdots \leq \mathsf{fdCat} \leq \infty\text{-}\mathsf{Cat}$$

so that the notation is actually nonambiguous. On the other hand, if you found the lack of details or examples for higher dimensions more annoying, then the next section is for you.

3.7 Details and Examples

Having completed the main induction, we are now in a position to give some higher dimensional examples and details. The first one is a friendly one, in dimension 2. Then I give is two foundational examples, in dimensions 2 and n. Probably the most important example, in dimension $n+1$, is the domain $\mathsf{Int}_n = \int^{A \in n\text{-}\mathsf{Cat}} (n+1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})$ of the covariant integral in dimension n as an $(n+1)$-functor, yet to come in chapter 4.

Details 3.7.1 (2-Dimensional Covariant Integral Categories)**.**

Let $A \xrightarrow{U_\Box} 2\text{-}\mathsf{Cat}$ be a 2-functor. In the 2-category $\int_A U$, the cells take the form

$$\begin{tikzcd} (a,u) \arrow[r, bend left, "{(f,p)}"] \arrow[r, bend right, "{(g,q)}"'] \arrow[r, phantom, "\Downarrow(m,x)"] & (b,v) \end{tikzcd}$$

where $a \in |A|$ and $u \in |U_a|$ (and likewise for (b,v)), where $a \xrightarrow{f} b$ in A and $U_f u \xrightarrow{p} v$ in U_b, as in the 1-dimensional case (and likewise for (g,q)), and where, in A,

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow m"] & b \end{tikzcd}$$

and in U_b,

$$\begin{tikzcd} U_f u \arrow[r, "p"] \arrow[d, "U_m u"'] \arrow[dr, phantom, "\Downarrow x"] & v \arrow[d, "="] \\ U_g u \arrow[r, "q"'] & v \end{tikzcd}$$

Composition along 1-cells is defined for

$$\begin{tikzcd} (a,u) \arrow[rr, "{(f,p)}"] \arrow[d, "="'] & \arrow[d, phantom, "\Downarrow(m{,}x)"] & (b,v) \arrow[d, "="] \\ (a,u) \arrow[rr, "{(g,q)}"] \arrow[d, "="'] & \arrow[d, phantom, "\Downarrow(l{,}y)"] & (b,v) \arrow[d, "="] \\ (a,u) \arrow[rr, "{(h,r)}"'] & & (b,v) \end{tikzcd}$$

by

$$(l, y) \cdot (m, x) = (l \cdot m, y(U_m u) \cdot x),$$

the picture in U_b being

$$\begin{tikzcd} U_f u \arrow[r, "p"] \arrow[d, "U_m u"'] \arrow[dr, phantom, "\Downarrow x"] & v \arrow[d, "="] \\ U_g u \arrow[r, "q"] \arrow[d, "U_l u"'] \arrow[dr, phantom, "\Downarrow y"] & v \arrow[d, "="] \\ U_h u \arrow[r, "r"'] & v \end{tikzcd}$$

which makes sense because

$$(U_l u)(U_m u) = (U_l \cdot U_m)u = (U_{l \cdot m} u).$$

Composition along 0-cells is defined for

$$\begin{tikzcd} (a,u) \arrow[r, bend left, "{(f,p)}"] \arrow[r, bend right, "{(g,q)}"'] \arrow[r, phantom, "\Downarrow(m{,}x)"] & (b,v) \arrow[r, bend left, "{(h,r)}"] \arrow[r, bend right, "{(k,s)}"'] \arrow[r, phantom, "\Downarrow(l{,}y)"] & (c,w) \end{tikzcd}$$

by

$$(h, r)(f, p) = (hf, r(U_h p)),$$

as in dimension 1, and by

$$(l, y)(m, x) = (lm, y(U_h x)),$$

because $h = \mathsf{dom}_1\, l$. The picture in U_c is

$$\begin{tikzcd}
U_{hf}u \arrow[r, "U_h p"] \arrow[d, "U_{hm}u"'] \arrow[dr, phantom, "\Downarrow U_h x"] & U_h v \arrow[r, "r"] \arrow[d, "="] & w \arrow[d, "="] \\
U_{hg}u \arrow[r, "U_h q"'] \arrow[d, "U_{lg}u"'] & U_h v \arrow[r, "r"] \arrow[d, "U_l v"'] \arrow[dr, phantom, "\Downarrow y"] & w \arrow[d, "="] \\
U_{kg}u \arrow[r, "U_k q"'] & U_k v \arrow[r, "s"'] & w
\end{tikzcd}$$

which makes sense because

$$(U_{lg}u)(U_{hm}u) = [U_{lg} \cdot U_{hm}]u = U_{lg \cdot hm}u = U_{lm}u.$$

The NE square commutes for obvious reasons, and the SW square is the naturality of U_l (from U_h to U_k) at $U_g u \xrightarrow{q} v$.

The identities are comparatively boring.

Example 3.7.2 (The Fundamental Groupoid)**.**

The fundamental groupoid $\Pi_1 X$ of a topological space X is a well-known 2-functor from Top to $\mathsf{Cat} \leq 2\text{-}\mathsf{Cat}$. The integral 2-category $\mathcal{P} = \int_{\mathsf{Top}} \Pi_1$ serves as a convenient domain for a fundamental group functor.

The cells of $\mathcal{P}$ are

$$\begin{tikzcd}
(X, x_0) \arrow[r, bend left, "{(f,\varphi)}"] \arrow[r, bend right, "{(g,\gamma)}"'] \arrow[r, phantom, "{\Downarrow([H],1_\varphi)}"] & (Y, y_0)
\end{tikzcd}$$

where in Top,

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow[H]"] & Y \end{tikzcd},$$

and $x_0 \in X = |\Pi_1 X|$, while in in $\Pi_1 Y$,

$$\begin{tikzcd} fx_0 \arrow[r, "\varphi"] \arrow[d, "H^{x_0}"'] \arrow[dr, phantom, "\Downarrow 1_\varphi"] & y_0 \arrow[d, "="] \\ gx_0 \arrow[r, "\gamma"'] & y_0 \end{tikzcd}$$

the 2-cell 1_φ being forced – as a 1-category, $\Pi_1 Y$ has only identity 2-cells. Here, $H^{x_0} = (\Pi_1[H])x_0$ is the path-homotopy class of $\langle H(t, x_0)\rangle_{t\in I}$ and we see from $\gamma H^{x_0} = \varphi$ that any loop in $\gamma H^{x_0}\varphi^{-1}$ bounds a disk in Y, a condition necessary for the 2-cell $([H], 1_\varphi)$ to exist in $\mathcal{P}$.

So the 0-cells of $\mathcal{P}$ are pointed spaces, but the 1-cells aren't just the basepoint-preserving maps. They are arbitrary continuous maps coupled with a sort of correctional path-class from the image of the domain basepoint to the codomain basepoint. (So the maps aren't really completely arbitrary, as both basepoints must lie in the same path-component of the codomain space.)

Now we can formulate a 2-functor $\mathcal{P} \xrightarrow{\pi_1} \lfloor \mathsf{Grp} \rfloor$ mapping every 1-cell of $\mathcal{P}$ to a group homomorphism, and taking every 2-cell to an identity. (So in the situation above, we will have $\pi_1(f, \varphi) = \pi_1(g, \gamma)$.) The role of the correctional path-homotopy class in every 1-cell is to disambiguate how to navigate the potential change in basepoint. So this 2-functor will neatly package the conventional functoriality of the fundamental group with the classical facts about change of basepoint.

Let $\pi_1(X, x_0)$ be the usual fundamental group, continuing with the typical cell of $\mathcal{P}$ above.

The group morphism $\pi_1(X, x_0) \xrightarrow{\pi_1(f,\varphi)} \pi_1(Y, y_0)$ is given on $\xi \in \pi_1(X, x_0)$ by $[\pi_1(f, \varphi)]\xi = \varphi(f\xi)\varphi^{-1}$, where $f\xi$ abbreviates $(\Pi_1 f)\xi$. The picture in $\Pi_1 Y$ is

$$y_0 \xrightarrow{\varphi^{-1}} fx_0 \xrightarrow{f\xi} fx_0 \xrightarrow{\varphi} y_0 \; .$$

It is a morphism: for $\zeta, \xi \in \pi_1(X, x_0)$ we have

$$\begin{aligned}
[\pi_1(f,\varphi)](\zeta\xi) &= \varphi(f(\zeta\xi))\varphi^{-1} && (\text{def. of } \pi_1(f,\varphi)) \\
&= \varphi(f\zeta)(f\xi)\varphi^{-1} && (\Pi_1 \text{ is a 2-functor}) \\
&= \varphi(f\zeta)\varphi^{-1}\varphi(f\xi)\varphi^{-1} && (\varphi^{-1}\varphi = 1_{fx_0}) \\
&= ([\pi_1(f,\varphi)]\zeta)([\pi_1(f,\varphi)]\xi) && (\text{def. of } \pi_1(f,\varphi))
\end{aligned}$$

To see that $\pi_1(f,\varphi) = \pi_1(g,\gamma)$, let $\xi \in \pi_1(X, x_0)$ and form in $\Pi_1 Y$,

$$\begin{tikzcd}
y_0 \arrow[r, "\varphi^{-1}"] \arrow[d, "="'] & fx_0 \arrow[r, "f\xi"] \arrow[d, "H^{x_0}"'] & fx_0 \arrow[r, "\varphi"] \arrow[d, "H^{x_0}"] & y_0 \arrow[d, "="] \\
y_0 \arrow[r, "\gamma^{-1}"'] & fx_0 \arrow[r, "g\xi"'] & fx_0 \arrow[r, "\gamma"'] & y_0
\end{tikzcd}$$

the left square commuting by the definition of the 2-cell $([H], 1_\varphi)$, the right square by the left square, and the middle square by the naturality of $\Pi_1[H]$ at ξ. The top row is $[\pi_1(f,\varphi)]\xi$ which equals the bottom row, $[\pi_1(g,\gamma)]\xi$.

To see that $\mathcal{P} \xrightarrow{\pi_1} \lfloor \mathsf{Grp} \rfloor$ is a 2-functor we have left only to check that it preserves the composition of 1-cells, so let $\xi \in \pi_1(X, x_0)$ and let

$$(X, x_0) \xrightarrow{(f,\varphi)} (Y, y_0) \xrightarrow{(g,\gamma)} (Z, z_0)$$

be composable arrows in $\mathcal{P}$. Compute that

$$\begin{aligned}
[\pi_1((g,\gamma)(f,\varphi))]\xi &= [\pi_1(gf,\gamma(g\varphi))]\xi && \text{(composition in } \mathcal{P}\text{)} \\
&= \gamma(g\varphi)(gf\xi)(\gamma(g\varphi))^{-1} && \text{(def. of } \pi_1\text{)} \\
&= \gamma(g\varphi)(gf\xi)(g\varphi^{-1})\gamma^{-1} && \text{(distribute inverse)} \\
&= \gamma(g(\varphi(f\xi)\varphi^{-1}))\gamma^{-1} && \text{(factor out } g/\Pi_1 \text{ is a 2-functor)} \\
&= [\pi_1(g,\gamma)][\varphi(f\xi)\varphi^{-1}] && \text{(def. of } \pi_1(g,\gamma)\text{)} \\
&= [\pi_1(g,\gamma)][\pi_1(f,\varphi)]\xi && \text{(def. of } \pi_1(f,\varphi)\text{)}
\end{aligned}$$

and

$$\begin{aligned}
[\pi_1 1_{(X,x_0)}]\xi &= [\pi_1(1_X,1_{x_0})]\xi && \text{(def. of } 1_{(X,x_0)}\text{)} \\
&= 1_{x_0}(1_X\xi)1_{x_0}^{-1} && \text{(def. of } \pi_1(1_X,1_{x_0})\text{)} \\
&= \xi && (\Pi_1 \text{ is an } n\text{-functor; arithmetitic in } \Pi_1 Y) \\
&= 1_{\pi_1(X,x_0)}\xi && \text{(action of } 1_{\pi_1(X,x_0)}\text{)}
\end{aligned}$$

Background 3.7.3 (Limits in n-Categories)**.**

For an index category J, a J-complete n-category C is an n-category together with a specified right n-adjoint $C^J \xrightarrow{\mathsf{Lim}} C$ to the diagonal (constant functor forming) n-functor $C \xrightarrow{\Delta} C^J$ via the isomorphism of $(n-1)$-categories

$$C^J(\Delta a, S) \xrightarrow[\approx]{\mathsf{lim}} C(a, \mathsf{Lim}\, S),$$

n-natural in both the object a of C and the functor $J \xrightarrow{S} C$. I write

$$\mathsf{proj}^S_\square = \mathsf{lim}^{-1}\, 1_{\mathsf{Lim}\, S}$$

(here $a = \mathsf{Lim}\, S$) for the counit. If you are familiar with limits in n-categories, skip to the example following.

In perhaps more familiar terms, the functors S are J-indexed systems in C and the k-transformations $\Delta a \xrightarrow[{[k]}]{p_\Box} S$ in C^J are k-cones from a to S: n-natural families of k-cells $\langle a \xrightarrow[{[k]}]{p_i} Si \rangle_{i \in J}$ of C. The n-universal property of proj^S states that, for every k-cone $\Delta a \xrightarrow[{[k]}]{p} S$ in C^J, $\mathsf{lim}\, p$ is the unique k-cell of C such that, in C^J,

$$\begin{tikzcd}
\Delta a \arrow[d, "\Delta\, \mathsf{lim}\, p"', "{[k]}"] \arrow[dr, "p", "{[k]}"'] & \\
\Delta\, \mathsf{Lim}\, S \arrow[r, "\mathsf{proj}^S"'] & S
\end{tikzcd}$$

i.e., in C,

$$\begin{tikzcd}
a \arrow[d, "\mathsf{lim}\, p"', "{[k]}"] \arrow[dr, "p_i", "{[k]}"'] & \\
\mathsf{Lim}\, S \arrow[r, "\mathsf{proj}_i^S"'] & Si
\end{tikzcd}$$

for every object i of J. The limit object is often written

$$\mathsf{Lim}\, S = \mathsf{Lim}\langle Si \rangle_{i \in J} = \mathsf{Lim}_{i \in J}\, Si.$$

In the most familiar terms, at $J = \{0, 1\}$, a J-complete n-category is one with all binary products, and $\mathsf{Lim}\, S_\Box = S_0 \times S_1$. At $n = 2$ (say in Cat or in Top), the 2-universal property of the product can be illuminated by exploring how two 2-cells $\begin{tikzcd} X \arrow[r, bend left, "F_i"] \arrow[r, bend right, "G_i"'] \arrow[r, phantom, "\Downarrow H_i"] & Y_i \end{tikzcd}$, $i = 0, 1$, give rise to a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "F_0 \dot{\times} F_1"] \arrow[r, bend right, "G_0 \dot{\times} G_1"'] \arrow[r, phantom, "\Downarrow H_0 \dot{\times} H_1"] & Y_0 \times Y_1 \end{tikzcd}$, writing $A_0 \dot{\times} A_1$ for $\mathsf{lim}\, A_\Box$. (Note: $F_0 \dot{\times} F_1$ is quite different from $F_0 \times F_1$: $\begin{tikzcd} X \times X \arrow[r, bend left, "F_0 \times F_1"] \arrow[r, bend right, "G_0 \times G_1"'] \arrow[r, phantom, "\Downarrow H_0 \times H_1"] & Y_0 \times Y_1 \end{tikzcd}$ – this is why lim must differ from Lim notationally.)

Example 3.7.4 (Lim as an n-Functor for Varied Indices)**.**

Let $\mathcal{J}$ be the full subcategory of Cat of all J for which a given n-category C is J-complete. Form $\mathcal{L}_C = \int_{J \in \overline{\mathcal{J}}} n\text{-}\mathsf{Cat}(J, C)$. Now, $\mathcal{L}_C$ is n-dimensional since each $n\text{-}\mathsf{Cat}(J, C)$ is n-dimensional, but in the first coordinate of any k-cell we find only identities when $k > 1$. We can construe limits in C as an n-functor $\mathcal{L}_C \xrightarrow{\mathsf{Lim}} C$, as follows.

In $\mathcal{L}_C$, the k-cells

$$(J, S) \xrightarrow[{[k]}]{(F, X_\square)} (K, T) \xrightarrow[{[k]}]{(G, Y_\square)} (L, U)$$

mean that in $\mathcal{J}$,

$$J \xleftarrow{F} K \xleftarrow{G} L$$

are 1-cells, while in $n\text{-}\mathsf{Cat}$,

$$\begin{tikzcd}
C \arrow[d, phantom, ""] & J \arrow[l, "S"'] \\
C \arrow[u, "="] \arrow[ur, phantom, "{[k{+}1]}\Downarrow X"] & K \arrow[l, "T"'] \arrow[u, "F"'] \\
C \arrow[u, "="] \arrow[ur, phantom, "{[k{+}1]}\Downarrow Y"] & L \arrow[l, "U"] \arrow[u, "G"']
\end{tikzcd}$$

with the composite $(G, Y)(F, X) = (FG, Y \cdot XG)$, as pictured.

We define $\mathsf{Lim}(J, S) = \mathsf{Lim}\, S$ and $\mathsf{Lim}(F, X) = \mathsf{lim}(X \cdot \mathsf{proj}^S F)$, a k-cell from $\mathsf{Lim}(J, S)$ to $\mathsf{Lim}(K, T)$ since $X \cdot \mathsf{proj}^S F$, in $n\text{-}\mathsf{Cat}$,

$$\begin{tikzcd}
K \arrow[r, "F"] \arrow[rr, bend left=60, "\Delta \mathsf{Lim}\, S"] \arrow[rr, bend right=60, "T"'] \arrow[r, phantom, shift right=6, "{[k{+}1]}\Downarrow X"] & J \arrow[r, bend left, "\Delta \mathsf{Lim}\, S"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow \mathsf{proj}^S"] & C
\end{tikzcd}$$

is a k-cone from $\mathsf{Lim}\, S$ to T.

This is n-functorial along 0-cells as

$$\begin{aligned}
\mathsf{Lim}\, 1_{(K,T)} &= \mathsf{Lim}(1_K, 1_T) && \text{(def. of } 1_{(K,T)}) \\
&= \mathsf{lim}(1_T \cdot \mathsf{proj}^T\, 1_K) && \text{(def. of } \mathsf{Lim}) \\
&= \mathsf{lim}\, \mathsf{proj}^T && \text{(identity laws)} \\
&= 1_{\mathsf{Lim}\, T} && \text{(def. of } \mathsf{proj}^T) \\
&= 1_{\mathsf{Lim}(K,T)} && \text{(def. of } \mathsf{Lim})
\end{aligned}$$

and, for every object i of J,

$$\begin{aligned}
&\mathsf{proj}_i^U\, \mathsf{Lim}[(G,Y)(F,X)] \\
&= \mathsf{proj}_i^U\, \mathsf{Lim}(FG, Y \cdot XG) && \text{(compose in } \mathcal{L}_C) \\
&= \mathsf{proj}_i^U\, \mathsf{lim}(Y \cdot XG \cdot \mathsf{proj}^S\, FG) && \text{(def. of } \mathsf{Lim}) \\
&= (Y \cdot XG \cdot \mathsf{proj}^S\, FG)i && (n\text{-U.P. of } \mathsf{proj}_i^U) \\
&= Y_i X_{Gi}\, \mathsf{proj}_{FGi}^S && \text{(def. of } \cdot \text{ in } n\text{-}\mathsf{Cat}) \\
&= Y_i\, \mathsf{proj}_{Gi}^T[\mathsf{lim}(X \cdot \mathsf{proj}^S\, F)] && (n\text{-U.P. of } \mathsf{proj}_{Gi}^T, \cdot \text{ in } n\text{-}\mathsf{Cat}) \\
&= [(Y \cdot \mathsf{proj}^T\, G)i][\mathsf{lim}(X \cdot \mathsf{proj}^S\, F)] && \text{(def. of } \cdot \text{ in } n\text{-}\mathsf{Cat}) \\
&= \mathsf{proj}_i^U[\mathsf{lim}(Y \cdot \mathsf{proj}^T\, G)][\mathsf{lim}(X \cdot \mathsf{proj}^S\, F)] && \text{(def. of } \mathsf{proj}_i^U) \\
&= \mathsf{proj}_i^U[\mathsf{Lim}(G,Y)][\mathsf{Lim}(F,X)] && \text{(def. of } \mathsf{Lim})
\end{aligned}$$

so that by the n-universal property (n-U.P.) of proj^U,

$$\mathsf{Lim}[(G,Y)(F,X)] = [\mathsf{Lim}(G,Y)][\mathsf{Lim}(F,X)].$$

It is n-functorial along k-cells, $k > 0$ as, for l-cells (F, X) and (F', X') to be compos-

able along a k-cell, we must have $F = F'$ by the form of $\mathcal{L}_C$ so that X and X' are composable along a k-cell in $n\text{-}\mathsf{Cat}$ and we find

$$\begin{aligned}
&\mathsf{Lim}[(F, X') \cdot_k (F, X)] \\
&= \mathsf{Lim}[(F, X' \cdot_k X)] && \text{(compose in } \mathcal{L}_C\text{)} \\
&= \mathsf{lim}((X' \cdot_k X) \cdot \mathsf{proj}^S F) && \text{(def. of } \mathsf{Lim}\text{)} \\
&= \mathsf{lim}((X' \cdot \mathsf{proj}^S F) \cdot_k (X \cdot \mathsf{proj}^S F)) && \text{(butterfly law)} \\
&= [\mathsf{lim}(X' \cdot \mathsf{proj}^S F)] \cdot_k [\mathsf{lim}(X \cdot \mathsf{proj}^S F)] && (\mathsf{lim} \text{ is an } (n-1)\text{-functor}) \\
&= [\mathsf{Lim}(F, X')] \cdot_k [\mathsf{Lim}(F, X)] && \text{(def. of } \mathsf{Lim}\text{)}
\end{aligned}$$

by dimensional interchange and the $(n-1)$-functoriality of of lim.

This explains (among other things) why pull-backs always map to direct products; take J to be the free category generated by

$$0 \longrightarrow \star \longleftarrow 1$$

to get a pull-back, and consider the inclusion F into J of $K = \{0, 1\}$, which gives a direct product. Now $\mathsf{Lim}(F, 1_T)$ is the usual map from pull-back $\mathsf{Lim}(J, S)$ to product $\mathsf{Lim}(K, T)$ where T restricts S.

3.8 Summary

Here, for reference, I gather the useful details emerging from the main construction.

Given n-functors and n-natural transformations

$$\begin{tikzcd}
A \arrow[r, "U_\square"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow P"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
A' \arrow[r, "U'_\square"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{B} \arrow[r, "V_\square"] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\Downarrow Q_\square"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
\overline{B'} \arrow[r, "V'_\square"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}
\qquad
\begin{tikzcd}
A'' \arrow[r, "U''_\square"] & n\text{-}\mathsf{Cat} \\
\overline{B''} \arrow[r, "V''_\square"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}$$

with F, G, A'' and B'' cells of n-Cat, we have all of the following:

Proposition 3.8.1 (Universes). *Both* $\int_A U$ *and* $\int^B V$ *lie in* n-Cat.

Proposition 3.8.2 (Object Sets). *Both* $|\int_A U| = \int_{|A|} |\Box| U = \int_{a\in|A|} |U_a|$ *and* $|\int^B V| = \int^{|B|} |\Box| V = \int_{b\in|B|} |V_b|$.

Proposition 3.8.3 (Hom $(n-1)$-Categories). *For all* $(a,u),(b,v) \in |\int_A U|$,

$$[\textstyle\int_A U][(a,u),(b,v)] = \int^{f\in A(a,b)} U_b(U_f u, v)$$

and for all $(a,u),(b,v) \in |\int^B V|$,

$$[\textstyle\int^B V][(a,u),(b,v)] = \int_{g\in B(a,b)} V_a(u, V_g v).$$

Proposition 3.8.4 (Dualizations). *Both*

$$\textstyle\int_A U = \overline{\int^{\overline{A}^*} \overline{\Box}^* U^*}^{\,*} = \overline{\int^{a\in\overline{A}^*} \overline{U_a}^*}^{\,*}$$

and

$$\textstyle\int^B V = \overline{\int_{\overline{B}^*} \overline{\Box}^* V^*}^{\,*} = \overline{\int_{b\in\overline{B}^*} \overline{V_b}^*}^{\,*}.$$

Proposition 3.8.5 (Arithmetic). *In* $\int_A U$*, when defined,*

$$(g,q)(f,p) = (gf, q(U_{\mathsf{dom}_1 g}\, p)),$$

and in $\int^B V$*, when defined,*

$$(g,q)(f,p) = (gf, (U_{\mathsf{cod}_1 f}\, q)p).$$

Proposition 3.8.6 (Actions of Integral Functors). *The* n*-functor* $\int_A U \xrightarrow{\int_F P} \int_{A'} U'$ *acts on cells by*

$$[\textstyle\int_F P](f,p) = (Ff, P_{\mathsf{cod}_0 f}\, p),$$

while the n-functor $\int^B V \xrightarrow{\int^G Q} \int^{B'} V'$ *acts on cells by*

$$[\int^G Q](g,q) = (Gg, Q_{\mathsf{dom}_0\, g}\, q).$$

Proposition 3.8.7 (Cartesian Isomorphisms)**.** *The isomorphisms of n-categories,*

$$[\int_{a\in A} U_a] \times [\int_{b\in A''} U''_b] \xrightarrow[\approx]{\sigma} \int_{(a,b)\in A\times A''} U_a \times U''_b$$

and

$$[\int^{a\in B} V_a] \times [\int^{b\in B''} V''_b] \xrightarrow[\approx]{\sigma} \int^{(a,b)\in B\times B''} V_a \times V''_b\ ,$$

are both given on cells by the formula

$$\sigma((a,u),(b,v)) = ((a,b),(u,v)).$$

3.9 Inflation and Truncation of Integrals

In this section I show that the integrals are compatible with the tower

$$0\text{-}\mathsf{Cat} \le 1\text{-}\mathsf{Cat} \le 2\text{-}\mathsf{Cat} \le \cdots \le \mathsf{fdCat} \le \infty\text{-}\mathsf{Cat}$$

so that the notation is not ambiguous in terms of the dimension, and the integral thus extends to fdCat and ∞-Cat.

Proposition 3.9.1 (Compatability with Truncation)**.** *For m-functors (not n-functors!)* $A \xrightarrow{U_\square} m\text{-}\mathsf{Cat}$ *and* $\overline{B} \xrightarrow{V_\square} m\text{-}\mathsf{Cat}$*, we have the equalities of n-categories:*

$$\lfloor \int_A U \rfloor_n = \int_{a\in\lfloor A\rfloor_n} \lfloor U_a \rfloor_n \text{ and } \lfloor \int^B V \rfloor_n = \int^{b\in\lfloor B\rfloor_n} \lfloor V_b \rfloor_n.$$

Proof. For $n = 0$, this follows by definition, the 0-truncation being the object set.

Inducting, for the covariant case, note the object sets coincide and for objects (a,u) and (b,v) so do the hom-sets, by induction

$$\begin{aligned}
\left\lfloor \int_A U \right\rfloor_n [(a,u),(b,v)] &= \left\lfloor \left[\int_A U\right] [(a,u),(b,v)] \right\rfloor_{n-1} \\
&= \left\lfloor \int^{f\in A(a,b)} U_b(U_f u, v) \right\rfloor_{n-1} \\
&= \int^{f\in \lfloor A(a,b)\rfloor_{n-1}} \lfloor U_b(U_f u, v)\rfloor_{n-1} \\
&= \int^{f\in \lfloor A\rfloor_n(a,b)} \lfloor U_b\rfloor_n (U_f u, v) \\
&= \left[\int_{a\in\lfloor A\rfloor_n} \lfloor U_a\rfloor_n\right] [(a,u),(b,v)]
\end{aligned}$$

The identities remain the same, as does the composition. To see this latter fact, look at the definition of the compositions, and note how each step is compatable with truncation; this is an easy but highly notational exercise.

For the contravariant case, by the covariant case,

$$\begin{aligned}
\left\lfloor \int^B V \right\rfloor_n &= \left\lfloor \overline{\int_{b\in\overline{B}^*} \overline{V_b}^*}^* \right\rfloor_n \\
&= \overline{\left\lfloor \int_{b\in\overline{B}^*} \overline{V_b}^* \right\rfloor_n}^* \\
&= \overline{\int_{b\in\lfloor\overline{B}^*\rfloor_n} \lfloor\overline{V_b}^*\rfloor_n}^* \\
&= \overline{\int_{b\in\overline{\lfloor B\rfloor_n}^*} \overline{\lfloor V_b\rfloor_n}^*}^* \\
&= \int^{b\in\lfloor B\rfloor_n} \lfloor V_b\rfloor_n
\end{aligned}$$

QED

Corollary 3.9.2. *For U and V ∞-category-valued ∞-functors, both $\int_A U$ and $\int^B V$*

are ∞-categories when interpretted as the limits of the towers

$$\cdots \leq \int_{a\in\lfloor A\rfloor_n} \lfloor U_a\rfloor_n \leq \int_{a\in\lfloor A\rfloor_{n+1}} \lfloor U_a\rfloor_{n+1} \leq \cdots$$

and

$$\cdots \leq \int^{b\in\lfloor B\rfloor_n} \lfloor V_b\rfloor_n \leq \int^{b\in\lfloor B\rfloor_{n+1}} \lfloor V_b\rfloor_{n+1} \leq \cdots.$$

(Since Cat_∞ is the limit of the truncations.)

Proposition 3.9.3 (Compatability with Inflation)**.** *For m-functors, $0 \leq m \leq n$ $A \xrightarrow{U_\Box} m\text{-}\mathsf{Cat}$ and $\overline{B} \xrightarrow{V_\Box} m\text{-}\mathsf{Cat}$, we have the equalities of n-categories:*

$$\lceil \int_A U \rceil_{n+1} = \int_{a\in\lceil A\rceil_{n+1}} \lceil U_a\rceil_{n+1} \text{ and } \lceil \int^B V \rceil_{n+1} = \int^{b\in\lceil B\rceil_{n+1}} \lceil V_b\rceil_{n+1}.$$

Proof. At $m = 0$ in the covariant case, the object-sets coincide and for objects (a, u) and (b, v), so do the hom-sets:

$$\lceil \int_A U \rceil[(a, u), (b, v)] = \{(f, p) | a \xrightarrow{f} b \text{ in } A, U_f \xrightarrow{p} v \text{ in } U_B\}$$

is either just $\{(1_a, 1_u)\}$ (when $(a, u) = (b, v)$) or $\emptyset$ (otherwise), as is

$$\{(f, p) | a \xrightarrow{f} b \text{ in } \lceil A\rceil, U_f \xrightarrow{p} v \text{ in } \lceil U_B\rceil\} = [\int_{a\in\lceil A\rceil} \lceil U_a\rceil][(a, u), (b, v)].$$

The identities are unchanged, as is the composition – there are only identities to compose. The contravariant case of $m = 0$ is similar.

The remainder of the proof is the same as that of the last proposition, changing floors to ceilings and incrementing every dimensional subscript by one. QED

Corollary 3.9.4. *For U and V finite-dimensional ∞-category-valued ∞-functors of finite-dimensional ∞-categories, the integrals $\int_A U$ and $\int^B V$ are finite dimensional when interpretted as the colimits of the towers*

$$\cdots \leq \int_{a\in\lfloor A\rfloor_n} \lfloor U_a\rfloor_n \leq \int_{a\in\lfloor A\rfloor_{n+1}} \lfloor U_a\rfloor_{n+1} \leq \cdots$$

and

$$\cdots \leq \int^{b \in \lfloor B \rfloor_n} \lfloor V_b \rfloor_n \leq \int^{b \in \lfloor B \rfloor_{n+1}} \lfloor V_b \rfloor_{n+1} \leq \cdots .$$

(Since fdCat is the colimit of the inflation functors. Note by finite dimensionality that these towers stabilize for some n at the colimit.)

3.10 Cell Structure of Integrals

The definition of the integrals given so far is an inductive one. While I tend to prefer the inductive perspective of the definition of higher categories by enrichment, there will always be those who prefer the cellular definition, in terms of n-graphs.

Here is the cellular structure for the integrals, if only to convince you you don't really want to know. (The following is occasionally useful, but better avoided where possible.)

Theorem 3.10.1 (Cell Criterion)**.** *For an n-functor $A \xrightarrow{U_\square} n\text{-}\mathsf{Cat}$, we have*

$$(f,p) \in \left| \left[\int_A U \right] [(a_i,u_i),(b_i,v_i)]_{i=0}^{k-1} \right|$$

exactly when

$$f \in |A(a_i,b_i)_{i=0}^{k-1}| \text{ and } p \in |U_{b_0}(x_i,y_i)_{i=0}^{k-1}|$$

where $a_k = b_k = f$, $w_0 c = U_c u_0$ and

$$w_{i+1}c = \begin{cases} v_{i+1} \cdot_i w_i c & i \text{ even} \\ w_i c \cdot_i u_{i+1} & i \text{ odd} \end{cases} \quad (i = 1,2,\ldots,k-1)$$

and

$$(x_i,y_i) = \begin{cases} (w_i a_{i+1}, v_i) & i \text{ even} \\ (u_i, w_i b_{i+1}) & i \text{ odd} \end{cases} \quad (i = 0,1,\ldots,k-1).$$

Likewise, for an n-functor $\overline{B} \xrightarrow{V_{\square}} n\text{-}\mathsf{Cat}$, we have

$$(f, p) \in \left| \left[\int^B V \right] [(a'_i, u'_i), (b'_i, v'_i)]_{i=0}^{k-1} \right|$$

exactly when

$$f \in |B(a'_i, b'_i)_{i=0}^{k-1}| \text{ and } p \in |V_{a'_0}(x'_i, y'_i)_{i=0}^{k-1}|$$

where $a'_k = b'_k = f$, $w'_0 c = V_c v'_0$ and

$$w'_{n+1} c = \begin{cases} w'_n c \cdot'_n u'_{n+1} & n \text{ even} \\ v'_{n+1} \cdot'_n w'_n c & n \text{ odd} \end{cases} \quad (i = 1, 2, \ldots, k-1)$$

and

$$(x'_n, y'_n) = \begin{cases} (u'_n, w'_n b'_{n+1}) & n \text{ even} \\ (w'_n a'_{n+1}, v'_n) & n \text{ odd} \end{cases} \quad (i = 0, 1, \ldots, k-1).$$

Proof. We induct, the base case being vaccuous at $k = k' = 0$ or easy at $k = k' = 1$, if you're suspicious of starting at zero.

Inductively ($k > 0$), the following conditions are equivalent: that

$$\begin{aligned} (f, p) &\in \left| \left[\int_A U \right] [(a_i, u_i), (b_i, v_i)]_{i=0}^{k-1} \right| \\ &= \left| \left[\int^{g \in A(a_0, b_0)} U_{b_0}(U_g u_0, v_0) \right] [(a_{i+1}, u_{i+1}), (b_{i+1}, v_{i+1})]_{i=0}^{k-2} \right| \end{aligned}$$

and, under the substitutions

$$B = A(a_0, b_0), V = \langle U_{b_0}(U_g u_0, v_0) \rangle^{g \in B}, a'_i = a_{i+1}, b'_i = b_{i+1},$$

$$u'_i = u_{i+1} \text{ and } v'_i = v_{i+1}$$

which yield

$$
\begin{aligned}
w_0'c &= V_c v_0' \\
&= [U_{b_0}(U_c u_0, v_0]v_1 \\
&= v_1 \cdot_0 U_c u_0 \\
&= v_1 \cdot_0 w_0 c \\
&= w_1 c
\end{aligned}
$$

so that, inductively

$$
w_{i+1}' = \begin{cases} w_i'c \cdot_i' u_{i+1}' = w_{i+1}c \cdot_{i+1} u_{i+2} = w_{i+2}c & i \text{ even} \\ v_{i+1}' \cdot_i' w_i'c = v_{i+2} \cdot_{i+1} w_{i+1}c = w_{i+2}c & i \text{ odd} \end{cases}
$$

and thus

$$
(x_i', y_i') = \begin{cases} (u_i', w_i'b_{i+1}') = (u_{i+1}, w_{i+1}b_{i+2}) = (x_{i+1}, y_{i+1}) & i \text{ even} \\ (w_i'a_{i+1}', v_i') = (w_{i+1}a_{i+2}, v_{i+1}) = (x_{i+1}, y_{i+1}) & i \text{ odd} \end{cases}
$$

that both

$$
\begin{aligned}
f &\in |A(a_i, b_i)_{i=0}^{k-1}| \\
&= |A(a_0, b_0)(a_{i+1}, b_{i+1})_{i=0}^{k-2}| \\
&= |B(a_i', b_i')_{i=0}^{k'-1}|
\end{aligned}
$$

and

$$
\begin{aligned}
p &\in |U_{b_0}(x_i, y_i)_{i=0}k-1| \\
&= |U_{b_0}(U_{a_1}u_0, v_0)(x_{i+1}, y_{i+1})_{i=0}^{k-2}| \\
&= |V_{a'_0}(x'_i, y'_i)_{i=0}^{k'-1}|
\end{aligned}
$$

proving the covariant case.

The contravariant case follows the covariant under the substitutions

$$A = \overline{B}^*, U = \overline{\Box}^* V^*,$$

$$a_i = b'_i, b_i = a'_i,$$

$$u_i = v'_i, \text{ and } v_i = u'_i:$$

that

$$
\begin{aligned}
(f, p) &\in \left| \left[\int^B V \right] [(a'_i, u'_i), (b'_i, v'_i)]_{i=0}^{k'-1} \right| \\
&= \left| \overline{\int_{\overline{B}^*} \overline{\Box}^* V^*}^* [(a'_i, u'_i), (b'_i, v'_i)]_{i=0}^{k'-1} \right| \\
&= \left| \left[\int_{b \in \overline{B}^*} \overline{V_b}^* \right] [(b'_i, v'_i), (a'_i, u'_i)]_{i=0}^{k'-1} \right| \\
&= \left| \left[\int_A U \right] [(a_i, u_i), (b_i, v_i)]_{i=0}^{k-1} \right|
\end{aligned}
$$

is logically equivalent to the conjunction of

$$
\begin{aligned}
f &\in |B(a'_i, b'_i)_{i=0}^{k'-1}| \\
&= |\overline{B}^*(b'_i, a'_i)_{i=0}^{k'-1}| \\
&= |A(a_i, b_i)_{i=0}^{k-1}|
\end{aligned}
$$

with

$$\begin{aligned}
p &\in |V_{a'_0}(x'_i, y'_i)_{i=0}^{k'-1}| \\
&= |\overline{V_{a'_0}}^*(y'_i, x'_i)_{i=0}^{k'-1}| \\
&= |U_{b_0}(x_i, y_i)_{i=0}^{k-1}|
\end{aligned}$$

because we have

$$\begin{aligned}
w_0 c &= U_c u_0 \\
&= \overline{V_c}^* v'_0 \\
&= V_c v'_0 \\
&= w'_0 c
\end{aligned}$$

and, as such, inductively

$$w_{i+1} c = \begin{cases} v_{i+1} \cdot_i w_i c = w'_i c \cdot'_i u'_{i+1} = w'_{i+1} c & i \text{ even} \\ w_i c \cdot_i u_{i+1} = v'_{i+1} \cdot'_i w'_i c = w'_{i+1} c & i \text{ odd} \end{cases}$$

so that

$$(x_i, y_i) = \begin{cases} (w_i a_{i+1}, v_i) = (w'_i b'_{i+1}, u'_i) = (y'_i, x'_i) & i \text{ even} \\ (u_i, w_i b_{i+1}) = (v'_i, w'_i a'_{i+1}) = (y'_i, x'_i) & i \text{ odd} \end{cases}$$

QED

3.11 Domains of Integration

There are many full sub-$(n+1)$-categories $\mathcal{C}$ of n-Cat to which the integrals restrict in the sense that if $A \xrightarrow{U_\Box} \mathcal{C}$ is an n-functor with $A \in |\mathcal{C}|$ then $\int_A U \in |\mathcal{C}|$ and if $\overline{B} \xrightarrow{V_\Box} \mathcal{C}$

is an n-functor with $B \in |\mathcal{C}|$ then $\int^B V \in |\mathcal{C}|$. Let's call such sub-$(n+1)$-categories domains of integration.

Examples of domains of integration include Mon, the category of monoids, Ord, the category of (pre)orders, and according to the following, also the categories Grpd of groupoids, Grp of groups and EqR of equivalence relations.

Lemma 3.11.1 (Isomorphism Criterion)**.** *Let A and B lie in n-Cat and let $A \xrightarrow{U_\Box} n\text{-}\mathsf{Cat}$ and $\overline{B} \xrightarrow{V_\Box} n\text{-}\mathsf{Cat}$ be n-functors.*

If the 1-cell (f,p) of $\int_A U$ is an isomorphism, then

$$(f,p)^{-1} = (f^{-1}, U_f^{-1} p^{-1}),$$

and conversely, if both f and p are isomorphisms, then (f,p) is an isomorphism.

Again, if the 1-cell (g,q) of $\int^B V$ is an isomorphism, then

$$(g,q)^{-1} = (g^{-1}, V_g^{-1} q^{-1}),$$

and conversely, if both g and q are isomorphisms, then (g,q) is an isomorphism.

Writing ΥC for the groupoid of isomorphisms in a category C, we have both

$$\Upsilon \int_A U = \int_{a \in \Upsilon A} \Upsilon U_a \text{ and } \Upsilon \int^B V = \int^{b \in \Upsilon B} \Upsilon V_b.$$

Proof. Let $(a,u) \xrightarrow{(f,p)} (b,v) \xrightarrow{(g,q)} (a,u)$ in $\int_A U$, and suppose $(g,q) = (f,p)^{-1}$. Thus both

$$(gf, q(U_g p)) = (g,q)(f,p) = 1_{(a,u)} = (1_a, 1_u)$$

and

$$(fg, p(U_f q)) = (f,p)(g,q) = 1_{(b,v)} = (1_b, 1_v).$$

From the first coordinates, we see that $g = f^{-1}$. From the second coordinates, we see that

$$1_u = q(U_g p) = q(U_{f^{-1}} p) = q(U_f^{-1} p)$$

and that $1_v = p(U_f q)$; applying $U_{f^{-1}} = U_f^{-1}$ to this,

$$\begin{aligned}
U_f^{-1} 1_v &= 1_{U_f^{-1} v} && \text{(functoriality)} \\
&= U_f^{-1}[p(U_f q)] && \text{(r.h.s.)} \\
&= [U_f^{-1} p][U_f^{-1} U_f q] && \text{(functoriality)} \\
&= (U_f^{-1} p) q && \text{(cancellation)}
\end{aligned}$$

so that $q = (U_f^{-1} p)^{-1} = U_f^{-1} p^{-1}$.

Conversely, should both f and p be invertible, with $(a, u) \xrightarrow{(f,p)} (b, v)$ in $\int_A U$ as above, note from $U_f u \xrightarrow{p} v$ in U_b that $v \xrightarrow{p^{-1}} U_f u$ lies in U_b so that $U_f^{-1} v \xrightarrow{U_f^{-1} p^{-1}} U_f^{-1} U_f u = u$ lies in U_a and hence $(b, v) \xrightarrow{(f^{-1}, U_f^{-1} p^{-1})} (a, u)$ lies in $\int_A U$. It is left only to compute the composites

$$\begin{aligned}
(f^{-1}, U_f^{-1} p^{-1})(f, p) &= (f^{-1} f, (U_f^{-1} p^{-1})(U_{f^{-1}} p)) && \text{(def. of composition)} \\
&= (1_a, (U_f^{-1} p^{-1})(U_f^{-1} p)) && \text{(cancellation, } U \text{ is a functor)} \\
&= (1_a, U_f^{-1}(p^{-1} p)) && (U_f^{-1} \text{ is a functor)} \\
&= (1_a, U_f^{-1} 1_{U_f u}) && \text{(cancellation; } p \text{ vetted above)} \\
&= (1_a, 1_{U_f^{-1} U_f u}) && (U_f^{-1} \text{ is a functor)} \\
&= (1_a, 1_u) && \text{(cancellation)} \\
&= 1_{(a,u)} && \text{(def. of } 1_{(a,u)})
\end{aligned}$$

and

$$\begin{aligned}
(f,p)(f^{-1},U_f^{-1}p^{-1}) &= (ff^{-1},p(U_fU_f^{-1}p^{-1})) && \text{(def. of composition)}\\
&= (1_b, pp^{-1}) && \text{(cancellation)}\\
&= (1_b, 1_v) && \text{(more cancellation)}\\
&= 1_{(b,v)} && \text{(def. of } 1_{(b,v)}\text{)}
\end{aligned}$$

It is now immediate that $\Upsilon \int_A U = \int_{a\in\Upsilon A} \Upsilon U_a$.

As for the contravariant case, observe that $\Upsilon\overline{C}^* = \overline{\Upsilon C}^*$ for all categories C. Therefore,

$$\begin{aligned}
\Upsilon \int^B V &= \Upsilon\overline{\int_{b\in\overline{B}^*} \overline{V_b}^*}^*\\
&= \overline{\Upsilon \int_{b\in\overline{B}^*} \overline{V_b}^*}^*\\
&= \overline{\int_{b\in\Upsilon\overline{B}^*} \Upsilon\overline{V_b}^*}^*\\
&= \overline{\int_{b\in\overline{\Upsilon B}^*} \overline{\Upsilon V_b}^*}^*\\
&= \int^{b\in\Upsilon B} \Upsilon V_b
\end{aligned}$$

so that (g,q) is invertible if and only if both g and q are invertible and the formula for $(g,q)^{-1}$ is immediate. QED

Writing $\mathsf{Cat}_{\mathcal{C}}$ for the full sub-$(n+2)$-category of $(n+1)\text{-}\mathsf{Cat} = (n\text{-}\mathsf{Cat})\text{-}\mathsf{Cat}$ of all $(n+1)$-categories whose hom n-categories lie in $\mathcal{C}$, we also see that whenever $\mathcal{C}$ is a domain of integration, so are $\mathsf{Cat}_{\mathcal{C}}$, $\mathsf{Cat}_{\mathsf{Cat}_{\mathcal{C}}}$, $\mathsf{Cat}_{\mathsf{Cat}_{\mathsf{Cat}_{\mathcal{C}}}}$, and so on. So, for example, the 3-category $\mathsf{Cat}_{\mathsf{Ord}}$ of ordered categories is a domain of integration. (After all, $\mathsf{Ord} \leq \mathsf{Cat}$ is already a 2-category and enrichment adds a dimension.)

Finally, let $\mathcal{C}$ and $\mathcal{D}$ be domains of integration. Let $\mathcal{E}$ be the full sub-$(n+1)$-category of $\mathcal{C}$ consisting of all $C \in |\mathcal{C}|$ whose truncations to dimension k lie in $\mathcal{D}$: $\lfloor C \rfloor_k \in |\mathcal{D}|$. By the compatability of the integral with truncation, we see $\mathcal{E}$ is also a domain of integration. So, for example, the categories of ordered groupoids, of n-categories with only one object and of 5-categories in which every 3-cell is invertible, are all domains of integration.

In short, domains of integration abound.

Chapter 4

Higher Functoriality

What we have accomplished in the main construction is to state the action of the co- and contravariant integration $(n+1)$-functors on 0- and 1-cells. Now we turn our attention to their action on k-cells.

4.1 The Higher Functoriality of the Covariant Integral

Let $\mathsf{Int}_n = \int^{A \in n\text{-}\mathsf{Cat}} (n+1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})$ be the category of "integrands;" this will be the domain for an $(n+1)$-functor $\mathsf{Int}_n \xrightarrow{\int_\Box} n\text{-}\mathsf{Cat}$ with $\int_\Box(A, U) = \int_A U$ for 0-cells $(A, U) \in |\,\mathsf{Int}_n\,|$, which is to say, pairs of an n-category $A \in |n\text{-}\mathsf{Cat}\,|$ and an n-functor $U_\Box \in |(n+1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})|$. (Since A is n-dimensional, any $(n+1)$-functor from A is actually an n-functor. As in the main construction, the presence of $n+1$ in the definition of Int_n is only necessary at $n = 0$.)

We have already defined $\int_\Box$ on 0-cells. For $(A, U), (B, V) \in \mathsf{Int}_n$, define $\int_\Box$ on hom-n-categories by taking

$$\begin{tikzcd}
\mathsf{Int}_n[(A,U),(B,V)] \arrow[d, "="] \\
{[\int^{A\in n\text{-}\mathsf{Cat}}(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})][(A,U),(B,V)]} \arrow[d, "="] \\
\int_{F\in n\text{-}\mathsf{Cat}(A,B)}(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VF] \arrow[d, "{[\int_\Box]_{(A,U),(B,V)}}"] \\
n\text{-}\mathsf{Cat}(\int_A U, \int_B V)
\end{tikzcd}$$

as the curry of the n-functor

$$\begin{tikzcd}
{[\int_{F\in n\text{-}\mathsf{Cat}(A,B)}(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VF]] \times [\int_{a\in A} U_a]} \arrow[d, "\sigma", "\approx"'] \\
\int_{(F,a)\in n\text{-}\mathsf{Cat}(A,B)\times A}[(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VF]\times U_a] \arrow[d, "\int_{b=Fa} E_{F,a}", "J="'] \\
\int_{b\in B} V_b
\end{tikzcd}$$

where $E_{F,a}(P_\Box, u) = P_a u$ for all $(P,u) \in |(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VF]\times U_a|$.

Verifying the n-naturality of $E_{F,a}$ means checking that, in $n\text{-}\mathsf{Cat}'$,

$$\begin{tikzcd}
(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VF]\times U_a \arrow[r, "E_{F,a}"] \arrow[d, "{[k]}", "(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VX]\times U_f"'] & V_{Fa} \arrow[d, "V_{X_f}", "{[k]}"'] \\
(n+1)\text{-}\mathsf{Cat}'(A,n\text{-}\mathsf{Cat})[U,VG]\times U_b \arrow[r, "E_{G,b}"'] & V_{Gb}
\end{tikzcd}$$

for every k-cell $(F,a)\xrightarrow[{[k]}]{(X_\Box,f)}(G,b)$ of $n\text{-}\mathsf{Cat}(A,B)\times A$, which is to say, every pair of a $(k+1)$-cell $A\xrightarrow[{[k+1]}]{X_\Box}B$ of $n\text{-}\mathsf{Cat}$ and a k-cell $a\xrightarrow[{[k]}]{f}b$ of A.

Using the n-naturality of P at f and the definition of X_f, in $n\text{-}\mathsf{Cat}$ and in B respectively:

$$\begin{tikzcd}
U_a \arrow[r, "P_a"] \arrow[d, "{[k]}", "U_f"'] & V_{Fa} \arrow[d, "V_{Ff}", "{[k]}"'] \\
U_b \arrow[r, "P_b"'] & V_{Fb}
\end{tikzcd}
\qquad
\begin{tikzcd}
Fa \arrow[d, "{[k]}", "Ff"'] \arrow[dr, "X_f", "{[k]}"'] & \\
Fb \arrow[r, "X_b"', "{[k]}"] & Gb
\end{tikzcd}$$

calculate

$$\begin{aligned}
V_{X_f} E_{F,a}(P,u) &= V_{X_f} P_a u && \text{(def. of } E_{F,a}) \\
&= V_{X_b(Ff)} P_a u && \text{(def. of } X_f) \\
&= V_{X_b} V_{Ff} P_a u && (V \text{ is an } n\text{-functor}) \\
&= V_{X_b} P_b U_f u && (n\text{-naturality of } P \text{ at } f) \\
&= ((VX \cdot P)b) U_f u && \text{(def. of } \cdot \text{ in } n\text{-}\mathsf{Cat}) \\
&= E_{G,b}(VX \cdot P, U_f u) && \text{(def. of } E_{G,b})
\end{aligned}$$

which is $E_{G,b}[(n{+}1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})[U, VX] \times U_f](P,u)$ by the actions of $(n{+}1)\text{-}\mathsf{Cat}(\Box, \Box)$ and U_f. Thus $\int_\Box$ is a $(n+1)\text{-}\mathsf{Cat}'$-graph morphism.

Chasing cells (F,P) of $\mathsf{Int}_n[(A,U),(B,V)]$ and (f,p) of $\int_A U$ we find that

$$[\textstyle\int_F P](f,p) = J\sigma[(F,P),(f,p)] = J[(F,f),(P,p)] = (Ff, P_{\mathsf{cod}_0 f}\, p),$$

as before.

It remains only to check that $\int_\Box$ respects identities of 0-cells and composition along 0-cells. Accordingly, let

$$\begin{tikzcd}
(A,U) \arrow[r, bend left, "{(F,P_\Box)}"] \arrow[r, bend right, "{(F',P'_\Box)}"'] \arrow[r, phantom, "{[k]\Downarrow M_\Box}"] & (B,V) \arrow[r, bend left, "{(G,Q_\Box)}"] \arrow[r, bend right, "{(G',Q'_\Box)}"'] \arrow[r, phantom, "{[k]\Downarrow L_\Box}"] & (C,W)
\end{tikzcd}$$

be cells in Int_n, noting that

$$(G,Q)(F,P) = (GF, QF \cdot P)$$

and

$$(L,Y)(M,X) = (LM, YF' \cdot X).$$

For any arbitrary cell (f,p) of $\int_A U$ we find that

$$[\textstyle\int_{1_A} 1_U](f,p) = (1_A f, (1_U b)p) = (f, 1_{U_b} p) = (f,p) = 1_{\int_A U}(f,p)$$

and

$$\begin{aligned}
\left[\int_G Q\right]\left[\int_F P\right](f,p) &= \left[\int_G Q\right](Ff, P_{\mathsf{cod}_0 f}\, p) && \left(\text{action of } \int_F P\right)\\
&= (GFf, Q_{\mathsf{cod}_0 Ff} P_{\mathsf{cod}_0 f}\, p) && \left(\text{action of } \int_G Q\right)\\
&= (GFf, Q_{F\,\mathsf{cod}_0 f} P_{\mathsf{cod}_0 f}\, p) && (\mathsf{cod}_0\, Ff = F\,\mathsf{cod}_0\, f)\\
&= (GFf, ((QF \cdot P)\,\mathsf{cod}_0\, f)p) && (\text{def. of } \cdot \text{ in } n\text{-}\mathsf{Cat})\\
&= \left[\int_{GF} QF \cdot P\right](f,p) && \left(\text{action of } \int_{GF} QF \cdot P\right)
\end{aligned}$$

so that $\int_\Box$ preserves composition along 0-cells when applied to 1-cells. When $\int_\Box$ is applied to k-cells, $k > 1$, it produces a k-cell of $n\text{-}\mathsf{Cat}$ which is completely determined by its action on 0-cells (a,u) of $\int_A U$. In this case we calculate

$$\begin{aligned}
\left[\int_L Y\right]\left[\int_M X\right](a,u) &= \left[\int_L Y\right](M_a, X_a u) && \left(\text{action of } \int_M X\right)\\
&= (L_{M_a}, Y_{F'a} X_a u) && \left(\text{action of } \int_L Y\right)\\
&= ((LM)a, [(YF' \cdot X)a]u) && (\text{def. of } \mathsf{eval}, \cdot \text{ in } n\text{-}\mathsf{Cat})\\
&= \left[\int_{LM} YF' \cdot X\right](a,u) && \left(\text{action of } \int_{LM} YF' \cdot X\right)
\end{aligned}$$

Therefore, the covariant n-dimensional integral is an $(n+1)$-functor of Int_n.

Details 4.1.1 (2-Naturality of Integral Transformations)**.**

At this stage, the primary reason to exhibit the 2-naturality of an integral transformation is to help the reader follow these recursive definitions by unrolling them a bit. A secondary reason is to showcase the efficiency of the integral: the n-naturality

we check here is guaranteed by the construction, and would be nontrivial to check directly.

Let's check the 2-naturality of the transformation in $2\text{-}\mathsf{Cat}$,

$$\begin{tikzcd}
\int_A U \arrow[r, bend left, "\int_F P"] \arrow[r, bend right, "\int_G Q"'] \arrow[r, phantom, "\Downarrow\int_M X"] & \int_B V
\end{tikzcd}$$

arising from the 2-cell of Int_2,

$$\begin{tikzcd}
(A, U_\square) \arrow[r, bend left, "(F{,}P_\square)"] \arrow[r, bend right, "(G{,}Q_\square)"'] \arrow[r, phantom, "\Downarrow(M_\square{,}X_\square)"] & (B, V_\square)
\end{tikzcd}$$

which means that, in $2\text{-}\mathsf{Cat}'$,

$$\begin{tikzcd}
A \arrow[r, "U"] \arrow[d, bend right=50, "G"'] \arrow[d, bend left=50, "F"] \arrow[d, phantom, "\Leftarrow M"] \arrow[dr, phantom, "\Downarrow P"] & 2\text{-}\mathsf{Cat} \arrow[d, "="] \\
B \arrow[r, "V"'] & 2\text{-}\mathsf{Cat}
\end{tikzcd}
\quad X\Rrightarrow \quad
\begin{tikzcd}
A \arrow[r, "U"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow Q"] & 2\text{-}\mathsf{Cat} \arrow[d, "="] \\
B \arrow[r, "V"'] & 2\text{-}\mathsf{Cat}
\end{tikzcd}$$

since $n = 2 > 0$.

Now, for $(a, u) \in |\int_A U|$, $[\int_M X](a, u) = (M_a, X_a u)$ so the 2-naturality we must check, for every 2-cell of $\int_A U$,

$$\begin{tikzcd}
(a, u) \arrow[r, bend left, "(f{,}p)"] \arrow[r, bend right, "(g{,}q)"'] \arrow[r, phantom, "\Downarrow(m{,}x)"] & (b, v)
\end{tikzcd},$$

is that in $\int_B V$,

$$\begin{tikzcd}
{[\int_F P](a, u)} \arrow[r, bend left, "{[\int_F P](f,p)}"] \arrow[r, bend right, "{[\int_F P](g,q)}"'] \arrow[r, phantom, "{\Downarrow[\int_F P](m,x)}"] \arrow[d, "{[\int_M X](a,u)}"'] & {[\int_F P](b, v)} \arrow[d, "{[\int_M X](b,v)}"] \\
{[\int_G Q](a, u)} \arrow[r, bend left, "{[\int_G Q](f,p)}"] \arrow[r, bend right, "{[\int_G Q](g,q)}"'] \arrow[r, phantom, "{\Downarrow[\int_G Q](m,x)}"] & {[\int_G Q](b, v)}
\end{tikzcd}$$

which after integration amounts, in $\int_B V$, to

$$
\begin{tikzcd}
(Fa, P_a u) \arrow[rr, bend left, "{(Ff, P_b p)}"] \arrow[rr, bend right, "{(Fg, P_b q)}"'] \arrow[rr, phantom, "{\Downarrow(Fm, P_b x)}"] \arrow[d, "{(M_a, X_a u)}"'] & & (Fb, P_b v) \arrow[d, "{(M_b, X_b v)}"] \\
(Ga, Q_a u) \arrow[rr, bend left, "{(Gf, Q_b p)}"] \arrow[rr, bend right, "{(Gg, Q_b q)}"'] \arrow[rr, phantom, "{\Downarrow(Gm, Q_b x)}"] & & (Gb, Q_b v)
\end{tikzcd}
$$

which in turn amounts, after composition, to the equality of

$$(M_b, X_b v)(Fm, P_b x) = (M_b(Fm), (X_b v)(V_{M_b} P_b x))$$

with

$$(Gm, Q_b x)(M_a, X_a u) = ((Gm)M_a, (Q_b x)(V_{Gf} X_a u)).$$

The equality of the first coordinates is just the 2-naturality of M at m, that in B,

$$
\begin{tikzcd}
Fa \arrow[rr, bend left, "Ff"] \arrow[rr, bend right, "Fg"'] \arrow[rr, phantom, "\Downarrow Fm"] \arrow[d, "M_a"'] & & Fb \arrow[d, "M_b"] \\
Ga \arrow[rr, bend left, "Gf"] \arrow[rr, bend right, "Gg"'] \arrow[rr, phantom, "\Downarrow Gm"] & & Gb
\end{tikzcd}
$$

while the equality of the second coordinates uses the 2-naturality of M at m again, the naturality of of the 2-transformation X at f, that in $2\text{-}\mathsf{Cat}$,

$$
\begin{tikzcd}
U_a \arrow[rr, bend left, "V_{M_a} P_a"] \arrow[rr, bend right, "Q_a"'] \arrow[rr, phantom, "\Downarrow X_a"] \arrow[d, "U_f"'] & & V_{Ga} \arrow[d, "V_{Gf}"] \\
U_b \arrow[rr, bend left, "V_{M_b} P_b"] \arrow[rr, bend right, "X_b"'] \arrow[rr, phantom, "\Downarrow X_b"] & & Gb
\end{tikzcd}
$$

and the 2-naturality of the transformation X_b (in the last diagram) at the 2-cell x of X_b:

$$\begin{tikzcd}
U_f u \arrow[r, "p"] \arrow[d, "U_m u"'] \arrow[dr, phantom, "\Downarrow x"] & v \arrow[d, "="] \\
U_g u \arrow[r, "q"'] & v
\end{tikzcd}$$

to justify what we have already proved abstractly, that, in V_{Gb},

$$\begin{tikzcd}
 & & V_{M_b}P_b v \arrow[drr, "="] & & \\
V_{M_b}P_bU_f u \arrow[urr, "V_{M_b}P_b p"] \arrow[drr, "V_{M_b}P_bU_g u"'] \arrow[d, "="'] \arrow[rrrr, phantom, "\Downarrow V_{M_b}P_b x"] & & & & V_{M_b}P_b v \arrow[dddd, "X_b v"] \\
V_{M_b}V_{Ff}P_a u \arrow[d, "="'] & & V_{M_b}P_bU_g u \arrow[urr, "V_{M_b}P_b q"'] & & \\
V_{Gf}V_{M_a}P_a u \arrow[d, "X_bU_f u"', "=V_{Gf}X_a u"] & & & & \\
V_{Gf}Q_a u \arrow[d, "="'] & & Q_b v \arrow[drr, "="] & & \\
Q_bU_f u \arrow[urr, "Q_b p"] \arrow[drr, "Q_bU_m u"'] \arrow[rrrr, phantom, "\Downarrow Q_b x"] & & & & Q_b v \\
 & & Q_bU_g u \arrow[urr, "Q_b q"'] & &
\end{tikzcd}$$

Whew! But remember, we don't have to check these details every time – they are among those guaranteed by construction. For future reference, note as above:

Proposition 4.1.2 (2-Cells of Int_n)**.** *The 2-cells of* Int_n

$$\begin{tikzcd}
(A, U_\square) \arrow[r, bend left, "{(F,P_\square)}"] \arrow[r, bend right, "{(G,Q_\square)}"'] \arrow[r, phantom, "{\Downarrow(M_\square,X_\square)}"] & (B, V_\square)
\end{tikzcd}$$

mean exactly that, in n-Cat',

$$\begin{tikzcd}[column sep=large]
A \arrow[r, "U"] \arrow[d, bend right, "G"'] \arrow[d, bend left, "F"] \arrow[d, phantom, "\Leftarrow M"] \arrow[dr, phantom, "\Downarrow P"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
B \arrow[r, "V"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}
\quad X \Rrightarrow \quad
\begin{tikzcd}[column sep=large]
A \arrow[r, "U"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow Q"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
B \arrow[r, "V"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}$$

4.2 The Higher Functoriality of the Contravariant Integral

Let $\mathsf{cInt}_n = [\int^{B \in n\text{-}\mathsf{Cat}^*}[(n+1)\text{-}\mathsf{Cat}'(\overline{B}, n\text{-}\mathsf{Cat})]^*]^*$ be the category of "cointegrands;" this will be the domain of the contravariant n-dimensional integral $(n+1)$-functor $\int^{\Box}$, which will be the composition, in $(n+1)\text{-}\mathsf{Cat}'$,

$$\begin{array}{c}
\mathsf{cInt}_n = [\int^{B \in n\text{-}\mathsf{Cat}^*}[(n+1)\text{-}\mathsf{Cat}'(\overline{B}, n\text{-}\mathsf{Cat})]^*]^* \\
\Big\downarrow {\scriptstyle [\int^{A=\overline{B}^*}[(n+1)\,\mathsf{Cat}'(\overline{B}^*, \overline{\Box}^*)]\Box^*_{\overline{B}, n\text{-}\mathsf{Cat}}]^*} \\
\mathsf{Int}_n^* = [\int^{A \in n\text{-}\mathsf{Cat}}(n+1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})]^* \\
\Big\downarrow {\scriptstyle [\int_{\Box}]^*} \\
n\text{-}\mathsf{Cat}^* \\
\Big\downarrow {\scriptstyle \overline{\Box}^*} \\
n\text{-}\mathsf{Cat}
\end{array}$$

Vetting this definition is an exercise in dualities. I shall need two temporary notations to do it. For the n-category obtained by reversing only the 2-cells of an n-category C, I shall write $\widetilde{C}$, and for the n-category obtained by reversing all k-cells of C, $k \geq 3$, I shall write $C^\star$. The integrand defining cInt_n makes sense as the composition, in $(n+1)\text{-}\mathsf{Cat}'$ of

$$\overline{n\text{-}\mathsf{Cat}^*} = \overline{\widetilde{n\text{-}\mathsf{Cat}}^\star} \xrightarrow{\overline{[\overline{\Box}]}^\star} n\text{-}\mathsf{Cat}^\star \xrightarrow{[(n+1)\text{-}\mathsf{Cat}'(\Box, n\text{-}\mathsf{Cat})]^\star} n\text{-}\mathsf{Cat}^\star \xrightarrow{\Box^*} n\text{-}\mathsf{Cat};$$

the first step is the $\overline{\Box}^\star$-image of the $(n+1)$-functor $\widetilde{n\text{-}\mathsf{Cat}} \xrightarrow{\overline{\Box}} n\text{-}\mathsf{Cat}$, so the whole process is $B \mapsto \overline{B} \mapsto (n+1)\text{-}\mathsf{Cat}'(\overline{B}, n\text{-}\mathsf{Cat}) \mapsto [(n+1)\text{-}\mathsf{Cat}'(\overline{B}, n\text{-}\mathsf{Cat})]^*$. The

substitution $A = \overline{B}^*$ indeed runs from $n\,\text{-}\mathsf{Cat}^*$ to $n\,\text{-}\mathsf{Cat}$ and the integrand

$$J_B = [(n+1)\,\text{-}\mathsf{Cat}'(\overline{B}^*, \overline{\Box}^*)]\Box^*_{\overline{B}, n\,\text{-}\mathsf{Cat}}$$

makes sense in $n\,\text{-}\mathsf{Cat}$, as

$$\begin{tikzcd}
{[(n+1)\,\text{-}\mathsf{Cat}'(\overline{B}, n\,\text{-}\mathsf{Cat})]^*} \arrow[r, "="] \arrow[d, "J_B"'] & \mathsf{Cat}'^{\star}_{n+1}(\overline{B}, n\,\text{-}\mathsf{Cat}) \arrow[d, "\Box^*_{\overline{B}, n\,\text{-}\mathsf{Cat}}"] \\
(n+1)\,\text{-}\mathsf{Cat}'(\overline{B}^*, n\,\text{-}\mathsf{Cat}) & (n+1)\,\text{-}\mathsf{Cat}'(\overline{B}^*, n\,\text{-}\mathsf{Cat}^*) \arrow[l, "(n+1)\,\text{-}\mathsf{Cat}'(\overline{B}^*, \overline{\Box}^*)"]
\end{tikzcd}$$

using the $(n{+}1)$-functor $n\,\text{-}\mathsf{Cat}^{\star} \xrightarrow{\overline{\Box}^*} n\,\text{-}\mathsf{Cat}$. Chasing a cell Y of $(n{+}1)\,\text{-}\mathsf{Cat}'(\overline{B}, n\,\text{-}\mathsf{Cat})$ we obtain the formula $J_B Y = \overline{\Box}^* Y^*$.

This definition will make sense if J_B is n-natural in B, and then, chasing an arbitrary cell (B, V) of cInt_n we find $\int^B V = \overline{\int_{\overline{B}^*} \overline{\Box}^* V^*}^*$ as before.

To see that J_B is n-natural in B, we must show that for any k-cell of $(n+1)\,\text{-}\mathsf{Cat}'$,

$$B \xrightarrow[{[k]}]{F} C\ ,$$

that, in $n\,\text{-}\mathsf{Cat}'$,

$$\begin{tikzcd}
{[(n+1)\,\text{-}\mathsf{Cat}'(\overline{C}, n\,\text{-}\mathsf{Cat})]^*} \arrow[r, "J_C"] \arrow[d, "{[(n+1)\,\text{-}\mathsf{Cat}'(\overline{F}, n\,\text{-}\mathsf{Cat})]^*}"'] & (n+1)\,\text{-}\mathsf{Cat}'(\overline{C}^*, n\,\text{-}\mathsf{Cat}) \arrow[d, "(n+1)\,\text{-}\mathsf{Cat}'(\overline{F}^*, n\,\text{-}\mathsf{Cat})"] \\
{[(n+1)\,\mathsf{Cat}'(\overline{B}, n\,\text{-}\mathsf{Cat})]^*} \arrow[r, "J_B"'] & (n+1)\,\text{-}\mathsf{Cat}'(\overline{B}^*, n\,\text{-}\mathsf{Cat})
\end{tikzcd}$$

so let X be any cell of $(n+1)\,\text{-}\mathsf{Cat}'(\overline{C}, n\,\text{-}\mathsf{Cat})$ and compute

$$\begin{aligned}
[(n+1)\,\text{-}\mathsf{Cat}'(\overline{F}^*, n\,\text{-}\mathsf{Cat})]J_C X &= [(n+1)\,\text{-}\mathsf{Cat}'(\overline{F}^*, n\,\text{-}\mathsf{Cat})][\overline{\Box}^* X^*] && \text{(action of } J_C\text{)} \\
&= \overline{\Box}^* X^* \overline{F}^* && \text{(action of } (n+1)\,\text{-}\mathsf{Cat}'(\Box, \Box)\text{)} \\
&= \overline{\Box}^* (X\overline{F})^* && (\Box^* \text{ is an } (n+1)\text{-functor}) \\
&= J_B(X\overline{F}) && \text{(action of } J_B\text{)} \\
&= J_B[(n+1)\,\text{-}\mathsf{Cat}'(\overline{F}, n\,\text{-}\mathsf{Cat})]^* X && \text{(action of } (n+1)\,\text{-}\mathsf{Cat}'(\Box, \Box)\text{)}
\end{aligned}$$

Example 4.2.1 (B-Grph is 2-Functorial in B.)**.**

For a natural transformation

$$\begin{tikzcd} B \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow X"] & C \end{tikzcd}$$

we can define a natural transformation

$$\begin{tikzcd} Gr_B \arrow[r, bend left, "Gr_S"] \arrow[r, bend right, "Gr_T"'] \arrow[r, phantom, "\Downarrow Gr_X"] & Gr_C \end{tikzcd}$$

by using the fact that B-$\mathsf{Grph} = \int^{V \in \mathsf{Set}} \mathsf{Cat}'(V \times V, B)$ in concert with the higher functoriality of the contravariant integral. Indeed, we take

$$S\text{-}\mathsf{Grph} = \int^{V=V} \mathsf{Cat}'(V \times V, S)$$

and

$$X\text{-}\mathsf{Grph} = \int^{V=V} \mathsf{Cat}'(V \times V, X)$$

by the classical result that these integrands are n-natural. This forshadows (add reference).

The resulting structure, for a morphism $\Gamma \xrightarrow{\Phi} \Delta$ in B-Grph, is that $|S\text{-}\mathsf{Grph}\,\Gamma| = |\Gamma|$, $(S\text{-}\mathsf{Grph}\,\Gamma)(a,b) = S(\Gamma(a,b))$, $(S\text{-}\mathsf{Grph}\,\Phi)_{a,b} = S\Phi_{a,b}$, $|S\text{-}\mathsf{Grph}\,\Phi| = |\Phi|$, $|X\text{-}\mathsf{Grph}\,\Gamma| = 1_{|\Gamma|}$ and

$$(S\text{-}\mathsf{Grph}\,\Gamma)(a,b) \xrightarrow{(X\text{-}\mathsf{Grph}\,\Gamma)_{a,b} = X_{\Gamma(a,b)}} (T\text{-}\mathsf{Grph}\,\Gamma)(a,b) \text{ in } C \text{ for } a, b \in |\Gamma|.$$

The 2-functoriality of $\mathsf{Cat}' \xrightarrow{\mathsf{Grph}} \mathsf{Cat}'$ is an artifact of the 2-functoriality of integration, for if F and G are k-cells ($k = 1, 2$) of Cat' which are composable along an

l-cell, then

$$\begin{aligned}(G \cdot_1 F)\text{-}\mathsf{Grph} &= \int^{V=V} \mathsf{Cat}'(V \times V, G \cdot_l F) && \text{(def. of -}\mathsf{Grph}\\ &= \int^{V=V} \mathsf{Cat}'(V \times V, G) \cdot_l \mathsf{Cat}'(V \times V, F) && (\mathsf{Cat}'(V \times V, \Box) \text{ is a 2-functor}\\ &= \left[\int^{V=V} \mathsf{Cat}'(V \times V, G)\right] \cdot_l \left[\int^{V=V} \mathsf{Cat}'(V \times V, F)\right] && \text{(def. of } \cdot_l \text{ in } \mathsf{cInt}_2\\ &= G\text{-}\mathsf{Grph} \cdot_l F\text{-}\mathsf{Grph} && \text{(def. of -}\mathsf{Grph}\end{aligned}$$

The number of details to check making such a definition without the aid of the higher functoriality of the integral is a good demonstration of the efficiency it affords.

4.3 An Isomorphism of Interest

Proposition 4.3.1 (There is only one integral)**.** *Recall the* $(n+1)$*-functor in the first step of the definition of the contravariant integral:*

$$\begin{array}{c}\mathsf{cInt}_n = [\int^{B \in n\text{-}\mathsf{Cat}^*} [(n+1)\text{-}\mathsf{Cat}'(\overline{B}, n\text{-}\mathsf{Cat})]^*]^* \\ \Big\downarrow{\scriptstyle \omega}\ {\scriptstyle [\int^{A=\overline{B}^*} [(n+1)\text{-}\mathsf{Cat}'(\overline{B}^*, \overline{\Box}^*)] \Box^*_{\overline{B}, n\text{-}\mathsf{Cat}}]^*} \\ \mathsf{Int}_n^* = [\int^{A \in n\text{-}\mathsf{Cat}} (n+1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})]^*\end{array}$$

Now, ω *is an isomorphism* $\mathsf{cInt}_n \approx \mathsf{Int}_n^*$ *of* $(n+1)$*-categories, and*

$$\begin{tikzcd} \mathsf{cInt}_n \arrow[r, "\int^{\Box}"] \arrow[d, "\omega"', "\approx"] & n\text{-}\mathsf{Cat} \arrow[d, "\approx"', "{[\overline{\Box}^*]^*}"] \\ \mathsf{Int}_n^* \arrow[r, "{[\int_{\Box}]^*}"'] & n\text{-}\mathsf{Cat}^* \end{tikzcd}$$

(Here $[\overline{\Box}^*]^*$ *is the* $\Box^*$*-image of the duality* $\overline{\Box}^*$*.)*

Proof. Both coordinates of the integral defining ω are isomorphisms, so by the functoriality of the contravariant integral and the isomorphism criterion, ω is invertible.

Moreover, the duality $\overline{\Box}^*$ is invertible, whence its $\Box^*$-image must be invertible. The final square in the proposition simply rephrases the earlier definition of $\int^{\Box}$ in terms of $\int_{\Box}$. QED

Remark: On cells (A, U) of Int_n, $(\omega^{-1})^*(A, U) = (\overline{A}^*, \overline{\Box}^* U^*)$, reflecting the omnipresence of this pattern throughout this work. That ω and $(\omega^{-1})^*$ reverse all cells above dimension 1 is perfect, since we always apply $\overline{\Box}^*$ to the resulting integral, flipping exactly those cells back in the right direction. This justifies obtaining cells of cInt_n or Int_n in this way.

Chapter 5

Embeddings and Projections

The material in this section is not used much in the sequel, but deserves an early exhibition for several reasons: it is helpful in picturing the general structure of integral categories, it connects with the historical interest in fibration, and it differentiates integration and fibration in dimensions beyond the second.

5.1 Embeddings

Proposition 5.1.1 (Embedding Criterion). *For*

$$(A, U_\square) \xrightarrow{(F, P_\square)} (B, V_\square) \text{ in } \mathsf{Int}_n$$

and

$$(C, W_\square) \xrightarrow{(G, Q_\square)} (D, X_\square) \text{ in } \mathsf{cInt}_n,$$

the n-functor $\int_F P$ is an embedding if both F and every P_a are embeddings, while similarly the n-functor $\int^G Q$ is an embedding if both G and every Q_c are embeddings.

Proof. At $n = 0$, let $(a, u), (b, v) \in \int_A U$. If $[\int_F P](a, u) = [\int_F P](b, v)$, that is, if $(Fa, P_a u) = (Fb, P_b v)$, then $Fa = Fb$, whence $a = b$ and $P_a u = P_b v = P_a v$, whence $u = v$. The same proof applies to $\int^G Q$.

For $n > 0$ we induct. In the covariant case, $|\int_F P|$ is injective by the 0-dimensional case, and the hom-$(n-1)$-functors

$$\begin{tikzcd}
{[\int_A U][(a,u),(b,v)]} \arrow[r, "{[\int_F P]_{(a,u),(b,v)}}"] \arrow[d, "="'] & {[\int_B V][(Fa, P_a u), (Fb, P_b v)]} \arrow[d, "="] \\
\int^{f \in A(a,b)} U_b(U_f u, v) \arrow[r, "{\int^{g = F_{a,b} f} [P_b]_{U_f u, v}}"'] & \int^{g \in B(Fa, Fb)} V_{Fb}(V_{Ff} P_a u, P_b v)
\end{tikzcd}$$

are all embeddings by induction, since every $F_{a,b}$ and every $[P_b]_{U_f u, v}$ are embeddings.

In the contravariant case, $\overline{G}^*$ and every $\overline{Q_c}^* = (\overline{\Box}^* Q^*)c$ are embeddings, so $\int^G Q = \overline{\int_{\overline{G}^*} \overline{\Box}^* Q^*}^*$ is an embedding. QED

Proposition 5.1.2 (Direct products are integrals of constants)**.** *For n-categories A and B,*

$$A \times B = \int_{a \in A} B = \int^{a \in A} B.$$

Proof. At $n = 0$,

$$\int_{a \in A} B = \int^{a \in A} B = \{(a, b) | a \in A, b \in B\} = A \times B.$$

For $n > 0$, in the covariant case, we have

$$|A \times B| = |A| \times |B| = \int_{a \in |A|} |B| = |\int_{a \in A} B|,$$

so the object sets coincide.

Note that the integrand $\langle B \rangle_{a \in A}$ satisfies

$$\langle B \rangle_{a \in A} f = 1_B$$

for every k-cell f of A with $k > 0$.

For objects (a, b) and (c, d), the induction hypothesis yields that the hom-$(n-1)$-categories

$$\begin{aligned}
[A \times B][(a,b),(c,d)] &= A(a,c) \times B(b,d) && \text{(def. of } A \times B\text{)} \\
&= \int^{f \in A(a,c)} B(b,d) && \text{(induction)} \\
&= \int^{f \in A(a,c)} B(1_B b, d) && \text{(identity law)} \\
&= \left[\int_{a \in A} B\right][(a,b),(c,d)] && \text{(def. of homs)}
\end{aligned}$$

are equal. Now, every identity $1_{(a,b)} = (1_a, 1_b)$ also agrees, so we are only left to check the composition along 0-cells. Accordingly, let

$$(a,b) \xrightarrow[{[k]}]{(f,g)} (a',b') \xrightarrow[{[k]}]{(f',g')} (a'',b'') \text{ in } \textstyle\int_{a \in A} B,$$

and calculate that, in $\int_{a \in A} B$,

$$(f', g')(f, g) = (f'f, g'(1_B g)) = (f'f, g'g)$$

as it is in $A \times B$.

In the contravariant case, we have simply

$$A \times B = \overline{\overline{A}^* \times \overline{B}^*}^* = \overline{\textstyle\int_{a \in \overline{A}^*} \overline{B}^*}^* = \textstyle\int^{a \in A} B.$$

QED

Remark: We recover the familiar fact that the direct product group (or monoid) is the semidirect product with trivial action.

Proposition 5.1.3 (Canonical Embeddings)**.** *For $(A, U_\square) \in |\,\mathsf{Int}_n\,|$ and any object a of A, we have the canonical embedding*

$$U_a \xrightarrow{\mathsf{emb}_a} \textstyle\int_A U\ ,$$

given on cells u of U_a by $\mathsf{emb}_a\, u = (a,u)$, where here a denotes the identity of the same dimension as u. Similarly, for $(B, V_\Box) \in |\,\mathsf{cInt}_n\,|$ and any object b of B, we have the canonical embedding

$$V_b \xrightarrow{\mathsf{emb}_b} \int^B V\ ,$$

given on cells v of V_b by $\mathsf{emb}_b\, v = (b,v)$, where here b denotes the identity of the right dimension.

Proof. Abbreviate $\mathbb{1}_{n\text{-}\mathsf{Cat}}$ to simply $\mathbb{1}$, and for objects x in an n-category X, let $\S x$ be the name $\mathbb{1} \longrightarrow n\text{-}\mathsf{Cat}$ of x, given on the unique object $\Box \in \mathbb{1}$ by $[\S x]\Box = x$. (In other words, let $\S$ be the canonical isomorphism from X to $X^{\mathbb{1}}$.)

Now, emb_a is the composition

$$U_a \xrightarrow{\approx} \mathbb{1} \times U_a \xrightarrow{=} \int_{x\in\mathbb{1}} U_a \xrightarrow{\int_{\S a} 1_{\S U_a}} \int_A U\,,$$

where $\approx$ is the canonical isomorphism, the equality comes from the last proposition, and the third embedding comes, by the embedding criterion, from the transformation

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\S U_a"] \arrow[d, "\S a"'] \arrow[dr, phantom, "\Downarrow 1_{\S U_a}"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ A \arrow[r, "U"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

reflecting that $U(\S a) = \S U_a$, since every n-functor from $\mathbb{1}$ is an embedding.

Likewise, emb_b is the composition

$$V_b \xrightarrow{\approx} \mathbb{1} \times V_b \xrightarrow{=} \int^{x\in\mathbb{1}} V_b \xrightarrow{\int^{\S b} 1_{\S V_b}} \int^B V\,,$$

where the third embedding comes, by the embedding criterion, from the transformation

$$\begin{tikzcd} \overline{\mathbb{1}} = \mathbb{1} \arrow[r, "\S V_b"] \arrow[d, "\overline{\S b}"'] \arrow[dr, phantom, "\Downarrow 1_{\S V_b}"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ \overline{B} \arrow[r, "V"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

which reflects that $V\overline{\S b} = \S V_b$. QED

Remark: The images of these canonical embeddings are called the fibers of the integral, and the cells contained in these fibers are called flat cells.

5.2 Projections

Proposition 5.2.1 (Canonical Projections)**.** *For* $(A, U_\Box) \in |\,\mathsf{Int}_n\,|$ *and* $(B, V_\Box) \in |\,\mathsf{cInt}_n\,|$*, we have canonical projection* n*-functors*

$$\textstyle\int_A U \xrightarrow{\mathsf{proj}_{A,U}} A \ \textit{ and } \int^B V \xrightarrow{\mathsf{proj}_{B,V}} B\,,$$

given in both cases on cells (f, p) *by* $\mathsf{proj}(f, p) = f$.

Proof. In the case $n = 0$ there is nothing to check.

For $n > 0$, in the covariant case, it is clear that $\mathsf{proj}_{A,U}$ preserves identities at 0-cells and composition along 0-cells. It remains only to see that it has hom-$(n-1)$-functors. Let (a, u) and (b, v) be objects of $\int_A U$. Now the hom-$(n-1)$-functor mapping

$$[\textstyle\int_A U][(a, u), (b, v)] = \int^{f \in A(a,b)} U_b(U_f u, v)$$

to $A(\mathsf{proj}_{A,U}(a, u), \mathsf{proj}_{A,U}(b, v)) = A(a, b)$ is just

$$[\mathsf{proj}_{A,U}]_{(a,u),(b,v)} = \mathsf{proj}_{A(a,b),\langle U_b(U_f u, v)\rangle^{f \in A(a,b)}}\,,$$

according to its required action on cells, and it is an $(n-1)$-functor by induction.

In the contravariant case, we have

$$\mathsf{proj}_{B,V} = \overline{\mathsf{proj}_{\overline{B}^*,\langle \overline{V_b}^* \rangle_{b \in \overline{B}^*}}}^*,$$

forced on us by the required action of $\mathsf{proj}_{B,V}$ on cells. QED

Proposition 5.2.2 (Naturality of the Projections)**.** *In* $(n + 1)$-Cat',

$$\begin{tikzcd} \mathsf{Int}_n \arrow[r, bend left, "\int_\square"] \arrow[r, bend right, "\mathsf{proj}"'] \arrow[r, phantom, "\Downarrow\mathsf{proj}_\square"] & n\text{-}\mathsf{Cat} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathsf{cInt}_n \arrow[r, bend left, "\int^\square"] \arrow[r, bend right, "\mathsf{proj}^{\circ\circ}"'] \arrow[r, phantom, "\Downarrow\mathsf{proj}_\square"] & n\text{-}\mathsf{Cat}. \end{tikzcd}$$

Proof. The bottom $(n+1)$-functors vet by the definitions of Int_n and cInt_n (4.1 and 4.2). The $(n+1)$-naturality of the covariant projector proj in dimension n just means, for every k-cell of Int_n,

$$(A,U) \xrightarrow[{[k]}]{(F,P)} (B,V)\ ,$$

that in n-Cat,

$$\begin{tikzcd} \int_A U \arrow[r, "\mathsf{proj}_{A,U}"] \arrow[d, "\int_F P"', "{[k]}"] & A \arrow[d, "{[k]}"', "F"] \\ \int_B V \arrow[r, "\mathsf{proj}_{B,V}"'] & B \end{tikzcd}$$

which is obvious: chasing a j-cell (f,p) of $\int_A U$ we have

$$\mathsf{proj}_{B,V}[\textstyle\int_F P](f,p) = \mathsf{proj}_{B,V}(Ff, P_{\mathsf{cod}_0 f}\,p) = Ff = F\,\mathsf{proj}_{B,V}(f,p).$$

The $(n+1)$-naturality of the contravariant projector proj in dimension n just means, for every k-cell of cInt_n,

$$(A,U) \xrightarrow[{[k]}]{(F,P)} (B,V)\ ,$$

that in n-Cat,

$$\begin{tikzcd} \int^A U \arrow[r, "\mathsf{proj}_{A,U}"] \arrow[d, "\int^F P"', "{[k]}"] & A \arrow[d, "{[k]}"', "F"] \\ \int^B V \arrow[r, "\mathsf{proj}_{B,V}"'] & B \end{tikzcd}$$

which is equally obvious. QED

5.3 Canonical Lifts

Certain cells play a large role in the definition of a fibration or 2-fibration. They are usually called cartesian cells, but I will call them canonical lifts. First, let's define them.

Proposition 5.3.1 (Canonical lifts exist)**.** *Let*

$$(A, U_\Box) \in |\,\mathsf{Int}_n\,| \text{ and } (B, V_\Box) \in |\,\mathsf{cInt}_n\,|.$$

Assume $k > 1$. For any k-cell of A, $a \xrightarrow[{[k]}]{f} b$ and any $u \in |U_a|$, there is a k-cell $\tilde{f} = \mathsf{lift}[f, (a, u), (A, U)]$ of $\int_A U$ with $\mathsf{dom}_0\, \tilde{f} = (a, u)$, given by $\tilde{f} = (f, 1_{U_f u})$. Likewise, for any k-cell of B, $a \xrightarrow[{[k]}]{g} b$ and any $v \in |U_b|$, there is a k-cell $\tilde{g} = \mathsf{lift}'[g, (b, v), (B, V)]$ of $\int^B V$ with $\mathsf{cod}_0\, \tilde{g} = (b, v)$, given by $\tilde{g} = (g, 1_{V_g v})$.

Proof. At $k = 1$, in the covariant case, the arrow $U_f u \xrightarrow{1_{U_f u}} U_f u$ of U_b shows that, in $\int_A U$, $(a, u) \xrightarrow{\tilde{f}} (b, U_f u)$. In particular, $\mathsf{dom}_0\, \tilde{f} = (a, u)$. In the contravariant case, the arrow $V_g v \xrightarrow{1_{V_g v}} V_g v$ of U_b shows that, in $\int^B V$, $(a, V_g v) \xrightarrow{\tilde{g}} (b, v)$. In particular, $\mathsf{cod}_0\, \tilde{g} = (b, v)$.

For $k > 1$, induct on $k + n$.

In the covariant case, consider the $(k-1)$-cell, by induction

$$\widetilde{\mathsf{cod}\, f} = \mathsf{lift}[\mathsf{cod}\, f, (a, u), (A, U)] = (\mathsf{cod}\, f, 1_{U_{\mathsf{cod}\, f} u}),$$

with $\mathsf{dom}_0\, \widetilde{\mathsf{cod}\, f} = (a, u)$ and write $(b, v) = \mathsf{cod}_0\, \widetilde{\mathsf{cod}\, f}$. Now induct in the hom-$(n-1)$-category

$$[\textstyle\int_A U][(a, u), (b, v)] = \int^{h \in A(a,b)} U_b(U_h u, v)$$

to find

$$\begin{aligned}
&\mathsf{lift}'[f, (\mathsf{cod}\, f, 1_{U_{\mathsf{cod}\, f} u}), (A(a,b), \langle U_b(U_h u, v)\rangle^{h\in A(a,b)})] \\
&= (f, 1_{[U_b(U_f u, v)] 1_{U_{\mathsf{cod}\, f} u}}) \\
&= (f, 1_{1_{U_{\mathsf{cod}\, f} u}(U_f u)}) \\
&= (f, 1_{U_f u}) \\
&= \tilde{f}
\end{aligned}$$

with $\mathsf{dom}_0\, \tilde{f} = (a,u)$ since $\tilde{f}$ lies in the hom-$(n-1)$-category $[\int_A U][(a,u),(b,v)]$.

For the contravariant case,

$$\tilde{g} = \mathsf{lift}[g, (b,v), (\overline{B}^*, \overline{\Box}^* V^*)] = (g, 1_{\overline{V_g}^* v}) = (g, 1_{V_g v})$$

has $\mathsf{dom}_0\, \tilde{g} = (b,v)$ in $\int_{\overline{B}^*} \overline{\Box}^* V^* = \overline{\int^B V}^*$, and hence $\mathsf{cod}_0\, \tilde{g} = (b,v)$ in $\int^B V$. QED

Proposition 5.3.2 (Canonical factorizations)**.** *Let $(A, U_\Box) \in |\,\mathsf{Int}_n\,|$ for $n \le 2$. Then every k-cell of of $\int_A U$, $k \ge 1$, has a canonical factorization as a composition of a canonical lift with a flat cell (one in the image of one of the canonical embeddings).*

Proof. I prove this for the case $k = n = 2$, since this proof contains the details of the cases $(k,n) \in \langle (1,1),(1,2),(2,2)\rangle$. The 2-cell of $\int_A U$,

$$\begin{tikzcd} (a,u) \arrow[r, bend left, "{(f,p)}"] \arrow[r, bend right, "{(g,q)}"'] \arrow[r, phantom, "\Downarrow(m,x)"] & (b,v) \end{tikzcd},$$

factors, using the canonical lifts $\tilde{f}, \tilde{g}$ and $\tilde{m}$ of f, g and m at (a,u), as

$$\begin{tikzcd} (a,u) \arrow[r, "\tilde{f}"] \arrow[d, "="'] \arrow[dr, phantom, "\Downarrow\tilde{m}"] & (b, U_f u) \arrow[r, "{(b,p)}"] \arrow[d, "{(b, U_m u)}"'] \arrow[dr, phantom, "\Downarrow(b,x)"] & (b,v) \arrow[d, "="] \\ (a,u) \arrow[r, "\tilde{g}"'] & (b, U_g u) \arrow[r, "{(b,q)}"'] & (b,v) \end{tikzcd}$$

QED

Remarks: The uniqueness of such factorizations is the universal property used in the less constructive definition of fibrations through lifting properties.

We see also, that in dimensions $n \leq 2$, every integral n-category is generated by its canonical lifts and its flat cells. In fact, by composing such diagrams one can easily discover relations sufficient to present these 2-categories in terms of these generators.

On the other hand, this situation fails miserably for $n = 3$ and beyond. In detail, for $(A, U_\Box) \in |\,\mathsf{Int}_3\,|$, at a 3-cell of $\int_A U$,

$$\begin{tikzcd} (a,u) \arrow[r, bend left, "{(f,p)}"] \arrow[r, bend right, "{(g,q)}"'] \arrow[r, phantom, "\Downarrow(m,x)"] & (b,v) \arrow[r, phantom, "(\mu,\chi)\Rrightarrow"] & (a,u) \arrow[r, bend left, "{(f,p)}"] \arrow[r, bend right, "{(g,q)}"'] \arrow[r, phantom, "\Downarrow(l,y)"] & (b,v) \end{tikzcd},$$

the flat cell for χ (rotating the picture by $\pi/2$ for typographic reasons) makes sense as

$$\begin{tikzcd} (b,v) \arrow[r, "="] & (b,v) & & (b,v) \arrow[r, "="] & (b,v) \\ (b,U_f u) \arrow[u, "{(b,p)}"] \arrow[r, bend left, "{(b,U_m u)}"] \arrow[r, bend right, "{(b,U_l u)}"'] \arrow[r, phantom, "\Downarrow(b,U_\mu u)"] \arrow[ur, phantom, "(b,x)\Rightarrow"] & (b,U_g u) \arrow[u, "{(b,q)}"'] \arrow[r, phantom, "(b,\chi)\Rrightarrow"] & & (b,U_f u) \arrow[u, "{(b,p)}"] \arrow[r, "{(b,U_l u)}"'] \arrow[ur, phantom, "(b,y)\Rightarrow"] & (b,U_g u) \arrow[u, "{(b,q)}"'] \end{tikzcd}$$

and as such is not even composable with the canonical lift of μ at (a,u), which makes sense, in terms of the canonical lifts of m,l,f and g at (a,u), as

$$\begin{tikzcd} (b,U_f u) \arrow[r, "{(b,U_m u)}"] & (b,U_g u) & & (b,U_f u) \arrow[r, bend left, "{(b,U_m u)}"] \arrow[r, bend right, "{(b,U_l u)}"'] \arrow[r, phantom, "\Downarrow(b,U_\mu u)"] & (b,U_g u) \\ (a,u) \arrow[u, "\tilde{f}"] \arrow[r, "="'] \arrow[ur, phantom, "\tilde{m}\Rightarrow"] & (a,u) \arrow[u, "\tilde{g}"'] \arrow[r, phantom, "\tilde{\mu}\Rrightarrow"] & & (a,u) \arrow[u, "\tilde{f}"] \arrow[r, "="'] \arrow[ur, phantom, "\tilde{l}\Rightarrow"] & (a,u) \arrow[u, "\tilde{g}"'] \end{tikzcd}$$

rendering a factorization of (μ, χ) through $\tilde{\mu}$ impossible.

5.4 Integral n-Functors and k-transformations

Definition 5.4.1. *For $(A, U), (B, V) \in |\,\mathsf{Int}_n\,|$, to say a k-cell (either an n-functor or a $(k+1)$-transformation) $\int_A U \xrightarrow[{[k]}]{\Phi} \int_B V$ is integral over a k-cell $A \xrightarrow[{[k]}]{F} B$ is to assert that, in n-Cat,*

$$\begin{tikzcd} \int_A U \arrow[r, "\Phi", "{[k]}"'] \arrow[d, "\mathsf{proj}"'] & \int_B V \arrow[d, "\mathsf{proj}"] \\ A \arrow[r, "{[k]}", "F"'] & B \end{tikzcd}$$

commutes. Likewise, for $(A, U), (B, V) \in |\,\mathsf{cInt}_n\,|$, to say a k-cell $\int^A U \xrightarrow[{[k]}]{\Phi} \int^B V$ is integral over a k-cell $A \xrightarrow[{[k]}]{F} B$ is to assert that, in n-Cat,

$$\begin{tikzcd} \int^A U \arrow[r, "\Phi", "{[k]}"'] \arrow[d, "\mathsf{proj}"'] & \int^B V \arrow[d, "\mathsf{proj}"] \\ A \arrow[r, "{[k]}", "F"'] & B \end{tikzcd}$$

commutes.

Example 5.4.2 (Integrals are integral)**.**

If $(A, U) \xrightarrow[{[k]}]{(F,P)} (B, V)$ in Int_n [cInt_n, respecively] then $\int_F P$ [$\int^F P$, respectively] is integral over F (5.2.2). We are about to see that these are the only n-functors and k-transformations which are integral.

It will follow that Int_n is isomorphic to the $(n+1)$-category of covariant integral n-categories, integral n-functors and integral k-transformations, and similarly for cInt_n. The situation is generalizes the short exact sequences characterizing the semidirect product of groups.

Theorem 5.4.3 (Integral n-functors are integrals)**.** *Let*

$$(A, U_\square), (B, V_\square) \in |\,\mathsf{Int}_n\,|$$

and let the n-functor $\int_A U \xrightarrow{\Phi} \int_B V$ be integral over $A \xrightarrow{F} B$. Define

$$(A, U) \xrightarrow{(F, P_\square)} (B, V)$$

in Int_n for all $a \in |A|$ by the universal property of embeddings; in n-Cat,

$$\begin{tikzcd} U_a \arrow[r, "\mathsf{emb}_a"] \arrow[d, "P_a"'] & \int_A U \arrow[r, "\mathsf{proj}"] \arrow[d, "\Phi"] & A \arrow[d, "F"] \\ V_{Fa} \arrow[r, "\mathsf{emb}_{Fa}"'] & \int_B V \arrow[r, "\mathsf{proj}"'] & B \end{tikzcd}$$

Then $\Phi = \int_F P$.

Likewise, let $(A, U_\square), (B, V_\square) \in |\,\mathsf{cInt}_n\,|$ and let the n-functor $\int^A U \xrightarrow{\Phi} \int^B V$ be integral over $A \xrightarrow{F} B$. Define

$$(A, U) \xrightarrow{(F, P_\square)} (B, V)$$

in cInt_n for all $a \in |A|$ by the universal property of embeddings; in n-Cat,

$$\begin{tikzcd} U_a \arrow[r, "\mathsf{emb}_a"] \arrow[d, "P_a"'] & \int^A U \arrow[r, "\mathsf{proj}"] \arrow[d, "\Phi"] & A \arrow[d, "F"] \\ V_{Fa} \arrow[r, "\mathsf{emb}_{Fa}"'] & \int^B V \arrow[r, "\mathsf{proj}"'] & B \end{tikzcd}$$

Then $\Phi = \int^F P$.

Proof. In both cases, the image of $\Phi\,\mathsf{emb}_a$ is $\Phi\,\mathsf{proj}_{A,U}^{-1}\langle a\rangle$ which lies in $\mathsf{proj}_{B,V}^{-1}\langle Fa\rangle$, since Φ is integral over F, which is the image of emb_{Fa}. Thus, in both cases, n-functors P_a are defined by the universal property of embeddings, since Φ maps the image of emb_a into the image of emb_{Fa}.

At $n = 0$, P is vacuously 0-natural and for $(a, u) \in \int_A U$,

$$\Phi(a,u) = (Fa, P_a u) = [\textstyle\int_F P](a,u),$$

the first coordinate since Φ is integral over F and the second because $u \in U_a$ and P_a is the restriction of Φ to the image of U_a.

Now let $n > 0$ and examine the covariant case. On 0-cells we have $|\Phi| = \int_{|F|} |\square| P = |\int_F P|$ by the 0-dimensional case. Let $(a,u),(b,v) \in |\int_A U|$. Now, by the definition of the projections, $\Phi_{(a,u),(b,v)}$ is integral over $F_{\mathsf{proj}(a,u),\mathsf{proj}(b,v)} = F_{a,b}$ so by induction we may form

$$\begin{tikzcd}
U_b(U_f u, v) \arrow[r, "\mathsf{emb}_f"] \arrow[d, "Q_f"'] & \int^{f \in A(a,b)} U_b(U_f u, v) \arrow[r, "\mathsf{proj}"] \arrow[d, "\Phi_{(a,u),(b,v)}"] & A(a,b) \arrow[d, "F_{a,b}"] \\
V_{Fb}(V_{Ff} P_a u, P_b v) \arrow[r, "\mathsf{emb}_{Ff}"'] & \int^{g \in B(Fa,Fb)} V_{Fb}(V_g P_a u, P_b v) \arrow[r, "\mathsf{proj}"'] & B(Fa, Fb)
\end{tikzcd}$$

where the middle column contains the hom-$(n-1)$-categories

$$[\textstyle\int_A U][(a,u),(b,v)] \text{ and } [\textstyle\int_B V][(Fa, P_a u),(Fb, P_b v)].$$

Now, $\mathsf{emb}_f = [\mathsf{emb}_b]_{U_f u, v}$ and $\mathsf{emb}_{Ff} = [\mathsf{emb}_{Fb}]_{V_{Ff} P_a u, P_b v}$ are restrictions, so $Q_f = [P_b]_{U_f u, v}$ is a restriction. By induction we have

$$\Phi_{(a,u),(b,v)} = \textstyle\int^{g = F_{a,b} f} Q_f = \int^{g = F_{a,b} f} [P_b]_{U_f u, v},$$

making $\Phi = \int_F P$ by definition, if only we can show that P is n-natural; even without the naturality of P we have the formula $\Phi(f,p) = (Ff, P_b p)$ on cells (f,p) of $[\int_A U][(a,u),(b,v)]$.

To see that, in n-Cat',

$$\begin{tikzcd}
A \arrow[r, "U"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow P"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\
B \arrow[r, "V"'] & n\text{-}\mathsf{Cat}
\end{tikzcd}$$

we must show, for every $(k+1)$-cell, $k > 1$ of A,

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "{[k+1]}\Downarrow\varphi"] & b \end{tikzcd}$$

that, in n-Cat,

$$\begin{tikzcd} U_a \arrow[r, "P_a"] \arrow[d, "U_\varphi"', "{[k]}"] & V_{Fa} \arrow[d, "{[k]}"', "V_{F\varphi}"] \\ U_b \arrow[r, "P_b"'] & V_{Fb} \end{tikzcd}$$

the case for 1-cells f of A being handled by the 2-cell 1_f.

Accordingly, let $u \xrightarrow[{[j]}]{p} v$ in U_a be an arbitrary cell, and let $m = \mathsf{max}\langle j, k\rangle$. It will suffice to check both that $P_b U_f p = V_{Ff} P_a p$ and that $P_b U_\varphi v = V_{F\varphi} P_a v$, since by the definition of evaluation, we have both that, in U_b,

$$\begin{tikzcd} U_f u \arrow[d, "U_f p"', "{[j]}"] \arrow[dr, "U_\varphi p"] \arrow[dr, phantom, "{[m]}"'] & \\ U_f v \arrow[r, "{[k]}", "U_\varphi v"'] & U_g v \end{tikzcd}$$

and that, in V_{Fb},

$$\begin{tikzcd} V_{Ff} P_a u \arrow[d, "V_{Ff} P_a p"', "{[j]}"] \arrow[dr, "V_{F\varphi} P_a p"] \arrow[dr, phantom, "{[m]}"'] & \\ V_{Ff} P_a v \arrow[r, "{[k]}", "V_{F\varphi} P_a v"'] & V_{Fg} P_a v \end{tikzcd}$$

so that then we will have

$$
\begin{aligned}
P_b U_\varphi p &= P_b[(U_\varphi v)(U_f p)] \\
&= [P_b U_\varphi v][P_b U_f p] \\
&= [V_{F\varphi} P_a v][V_{Ff} P_a p] && \text{(we will show)} \\
&= V_{F\varphi} P_a p
\end{aligned}
$$

for every p so that $P_b U_\varphi = V_{F\varphi} P_a$, which is the n-naturality of P we seek.

To see that $P_b U_f p = V_{Ff} P_a p$ consider the cells of $\int_A U$,

$$(a, u) \xrightarrow[{[j]}]{(a,p)} (a, v) \xrightarrow{(f, 1_{U_f v})} (b, v) \; .$$

(The 1-cell is the canonical lift of f at (a, v).) Now,

$$(f, U_f p) = (f, 1_{U_f v}(U_f p)) = (f, 1_{U_f v})(a, p),$$

so that in $\int_B V$,

$$
\begin{aligned}
(Ff, P_b U_f p) &= \Phi(f, U_f p) && \text{(action of } \Phi) \\
&= [\Phi(f, 1_{U_f v})][\Phi(a, p)] && (\Phi \text{ is an } n\text{-functor; factor } (f, U_f p)) \\
&= (Ff, P_b 1_{U_f v})(Fa, P_a p) && \text{(action of } \Phi) \\
&= (Ff, 1_{P_b U_f v}(V_{Ff} P_a p)) && \text{(def. of composition)} \\
&= (Ff, V_{Ff} P_a p) && \text{(identity law)}
\end{aligned}
$$

The desired equality is obtained by extracting the second coordinates.

To see that $P_b U_\varphi v = V_{F\varphi} P_a v$, consider the $(n-1)$-natural Q coming from the hom-$(n-1)$-category of $\int_A U$ containing the 1-cell $(a, u) \xrightarrow{(g, 1_{U_g u})} (b, U_g u)$. The $(n-1)$-naturality of Q at φ gives us that, in $(n-1)\,\text{-}\mathsf{Cat}$,

$$\begin{tikzcd}
U_b(U_g u, U_g u) \arrow[r, "Q_g"] \arrow[d, "U_b(U_\varphi u, U_g u)"', "{[k]}"] & V_{Fb}(F_{Fg} P_a u, P_b U_g u) \arrow[d, "{[k]}"', "V_{Fb}(V_{F\varphi} P_a u, P_b U_g u)"] \\
U_b(U_f u, U_g u) \arrow[r, "Q_f"'] & V_{Fb}(V_{Ff} P_a u, P_b U_g u)
\end{tikzcd}$$

(here, $v = U_g u$) so that,

$$\begin{aligned}
P_b U_\varphi u &= [P_b]_{U_f u, U_g u} U_\varphi u && \text{(restrict } P_b \text{ to hom)}\\
&= Q_f U_\varphi u && (Q_f \text{ restricts } P_b)\\
&= Q_f [1_{U_g u}(U_\varphi u)] && \text{(identity law)}\\
&= Q_f [U_b(U_\varphi u, U_g u)] 1_{U_g u} && \text{(action of } U_b(\square, \square))\\
&= [V_{Fb}(V_{F\varphi} P_a u, P_b U_g u)] Q_g 1_{U_g u} && ((n-1)\text{-naturality of } Q \text{ at } \varphi)\\
&= [V_{Fb}(V_{F\varphi} P_a u, P_b U_g u)] [P_b]_{U_g u, U_g u} 1_{U_g u} && (Q_g \text{ restricts } P_b)\\
&= [V_{Fb}(V_{F\varphi} P_a u, P_b U_g u)] P_b 1_{U_g u} && \text{(unrestrict } P_b \text{ from hom)}\\
&= [V_{Fb}(V_{F\varphi} P_a u, P_b U_g u)] 1_{P_b U_g u} && (P_b \text{ is an } n\text{-functor})\\
&= [V_{Fb}(V_{F\varphi} P_a u, P_b U_g u)] 1_{V_{Fg} P_a u} && (n\text{-naturality of } P \text{ at } g)\\
&= 1_{V_{Fg} P_a u}(V_{F\varphi} P_a u) && \text{(action of } V_{Fb}(\square, \square))\\
&= V_{F\varphi} P_a u && \text{(identity law)}
\end{aligned}$$

concluding the covariant case.

Now the contravariant case will be easy. Since $\overline{\Phi}^*$ is integral over $\overline{F}^*$, by the covariant case we have

$$\begin{tikzcd}
\overline{U_a}^* \arrow[r, "\mathsf{emb}_a"] \arrow[d, "Q_a"'] & \int_{a \in \overline{A}^*} \overline{U_a}^* \arrow[r, "\mathsf{proj}"] \arrow[d, "\overline{\Phi}^*"] & \overline{A}^* \arrow[d, "\overline{F}^*"] \\
\overline{V_{Fa}}^* \arrow[r, "\mathsf{emb}_{Fa}"'] & \int_{b \in \overline{B}^*} \overline{V_b}^* \arrow[r, "\mathsf{proj}"'] & \overline{B}^*
\end{tikzcd}$$

and so $P = \overline{\square}^* Q^*$ is n-natural, and

$$\Phi = \overline{\overline{\Phi}^*}^* = \overline{\textstyle\int_{\overline{F}^*} Q}^* = \overline{\textstyle\int_{\overline{F}^*} \overline{\square}^* P^*} = \textstyle\int^F P,$$

concluding the proof. QED

Theorem 5.4.4 (Integral k-transformations are integrals)**.** *Let $(A, U_\square), (B, V_\square) \in \mathsf{Int}_n$ and assume the k-transformation $\int_A U \xrightarrow[{[k+1]}]{\Gamma_\square} \int_B V$ in n-Cat is integral over the k-transformation $A \xrightarrow[{[k+1]}]{M} B$ in n-Cat. Then there is a $(k+1)$-cell $(A, U) \xrightarrow[{[k+1]}]{(M_\square, X_\square)} (B, V)$ in Int_n, given by $(M_a, X_a u) = \Gamma_{a,u}$ for all $a \in |A|$ and $u \in |U_a|$, such that $\Gamma = \int_M X$.*

Likewise, if $(A, U_\square), (B, V_\square) \in \mathsf{cInt}_n$ and if the k-transformation $\int^A U \xrightarrow[{[k+1]}]{\Gamma_\square} \int^B V$ in n-Cat is integral over the k-transformation $A \xrightarrow[{[k+1]}]{M} B$ in n-Cat, then there is a $(k+1)$-cell $(A, U) \xrightarrow[{[k+1]}]{(M_\square, X_\square)} (B, V)$ in cInt_n, given by $(M_a, X_a u) = \Gamma_{a,u}$ for all $a \in |A|$ and $u \in |U_a|$, such that $\Gamma = \int^M X$.

Proof. For the covariant case, the first thing to note is that the definition makes sense. Indeed, the first coordinate of $\Gamma_{a,u}$ must be M_a since Γ is integral over M, and the second coordinate is what we seek to define. If this definition for X is suitably natural, we'll be done, since Γ is completely determined by it action on objects (a, u) of $\int_A U$ and $[\int_M X](a, u) = (M_a, X_a u)$.

Let $\Phi = \mathsf{dom}_1 \Gamma$ and $\Psi = \mathsf{cod}_1 \Gamma$. Then Φ is integral over the n-functor $F = \mathsf{dom}_1 M$, and Ψ is integral over $G = \mathsf{cod}_1 M$. By the previous theorem, we have both $P_\square$ and $Q_\square$ so that $(F, P), (G, Q) \in |\,\mathsf{Int}_n[(A, U), (B, V)]|$ enjoy both $\Phi = \int_F P$ and $\Psi = \int_G Q$.

Now, X makes sense since $\Phi(a, u) = (Fa, P_a u)$ and $\Psi(a, u) = (Ga, Q_a u)$ and therefore

$$(Fa, P_a u) \xrightarrow[{[k]}]{\Gamma_{a,u} = (M_a, X_a u)} (Ga, Q_a u) \text{ in } \textstyle\int_B V,$$

which means that in V_{Ga},

$$V_{M_a}P_a u \xrightarrow[{[k]}]{X_a u} Q_a u\,.$$

We need to show that X is a n-natural k-transformation from $VM \cdot P$ to Q, i.e. that for every 1-cell $a \xrightarrow{f} b$ of A that, in n-Cat,

$$\begin{tikzcd}
U_a \arrow[r, bend left, "V_{M_a}P_a"] \arrow[r, bend right, "Q_a"'] \arrow[r, phantom, "{[k]}\Downarrow X_a"] \arrow[d, "U_f"'] & V_{Ga} \arrow[d, "V_{Gf}"] \\
U_b \arrow[r, bend left, "V_{M_b}P_b"] \arrow[r, bend right, "Q_b"'] \arrow[r, phantom, "{[k]}\Downarrow X_b"] & V_{Gb}
\end{tikzcd}$$

in symbols, that $X_b U_f = V_{Gf} X_a$.

Let $u \in |U_a|$ and consider the canonical lift $(f, 1_{U_f u})$ of f at (a, u), which has images $\Phi(f, 1_{U_f u}) = (Ff, P_b 1_{U_f u}) = (Ff, 1_{P_b U_f u})$ and $\Psi(f, 1_{U_f u}) = (Gf, 1_{Q_b U_f u})$. The naturality of Γ at $(f, 1_{U_f u})$ gives that, in $\int_B V$,

$$\begin{tikzcd}
(Fa, P_a u) \arrow[r, "{(Ff, 1_{P_b U_f u})}"] \arrow[d, "{(M_a, X_a u)}"', "{[k]}"] & (Fb, P_b U_f u) \arrow[d, "{[k]}"', "{(M_b, X_b U_f u)}"] \\
(Ga, Q_a u) \arrow[r, "{(Gf, 1_{Q_b U_f u})}"'] & (Gb, Q_b U_f u)
\end{tikzcd}$$

whence

$$\begin{aligned}
(M_b(Ff), X_b U_f u) &= (M_b(Ff), (X_b U_f u)(V_{M_b} 1_{P_b U_f u})) \\
&= (M_b, X_b U_f u)(Ff, 1_{P_b U_f u}) \\
&= (Gf, 1_{Q_b U_f u})(M_a, X_a u) \\
&= ((Gf)M_a, 1_{Q_b U_f u}(V_{Gf} X_a u)) \\
&= ((Gf)M_a, V_{Gf} X_a u),
\end{aligned}$$

proving that $X_b U_f u = V_{Gf} X_a u$ for every object u of U_a, and hence that the k-transformations $X_b U_f$ and $V_{Gf} X_a$ are equal, completing the proof for the covariant

case.

For the contravariant case, note that $\overline{\Gamma}^*$ is integral over $\overline{M}^*$ to obtain a $(k+1)$-cell Y of Int_n such that $\overline{\Gamma}^* = \int_{\overline{M}^*} Y$. Putting $X = \overline{\Box}^* Y^*$, we have that

$$\Gamma = \overline{\overline{\Gamma}^*}^* = \overline{\int_{\overline{M}^*} Y}^* = \overline{\int_{\overline{M}^*} \overline{\Box}^* X^*}^* = \int^M X.$$

QED

Chapter 6

Limit Properties

So far we have seen that integration produces n-categories, n-functors and n-natural k-transformations. Now we will see that, under appropriate conditions, they are good ones.

6.1 Background and Notation

The following notations were introduced in background 3.7.3. Recall, for an index category J, a J-complete n-category C is an n-category together with a specified right n-adjoint $C^J \xrightarrow{\mathsf{Lim}} C$ to the diagonal (constant-forming) n-functor $C \xrightarrow{\Delta} C^J$ via the isomorphism of $(n-1)$-categories

$$C^J(\Delta a, S) \xrightarrow[\approx]{\mathsf{lim}} C(a, \mathsf{Lim}\, S),$$

n-natural in both the object a of C and the functor $J \xrightarrow{S} C$. I write

$$\mathsf{proj}^S_\square = \mathsf{lim}^{-1}\, 1_{\mathsf{Lim}\, S}$$

(at $a = \mathsf{Lim}\, S$) for the counit. These notations are unusually well-chosen, so I repeat them here – background 3.7.3 may help connect this to other expositions of categorical

limits.

Let $D \xrightarrow{F} C$ be an n-functor between J-complete n-categories and let $J \xrightarrow{S} D$ be a functor. The canonical arrow $F\,\mathsf{Lim}\,S \xrightarrow{\mathsf{can}_F^S} \mathsf{Lim}\,FS$ of C is $\mathsf{can}_F^S = \mathsf{lim}\,F\,\mathsf{proj}^S$:

$$\begin{tikzcd} F\,\mathsf{Lim}\,S \arrow[rr, "\mathsf{can}_F^S"] \arrow[dr, "F\,\mathsf{proj}_i^S"'] & & \mathsf{Lim}\,FS \arrow[dl, "\mathsf{proj}_i^{FS}"] \\ & Si & \end{tikzcd}$$

for every $i \in |J|$. It is not hard to see that can_F^S is n-natural in both S and F: if

$$\begin{tikzcd} J \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "{[k{+}1]}\Downarrow X_\square"] & D \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "{[k{+}1]}\Downarrow Y_\square"] & C \end{tikzcd} \text{ in } n\text{-}\mathsf{Cat},$$

then in D both

$$\begin{tikzcd} F\,\mathsf{Lim}\,S \arrow[rr, "\mathsf{can}_F^S"] \arrow[dr, "F\,\mathsf{proj}^S\,i"] \arrow[ddd, "F\,\mathsf{Lim}\,X"', "{[k]}"] & & \mathsf{Lim}\,FS \arrow[dl, "\mathsf{proj}_i^{FS}"'] \arrow[ddd, "{[k]}"', "\mathsf{Lim}\,FX"] \\ & FSi \arrow[d, "{[k]}"', "FX_i"] & \\ & FTi & \\ F\,\mathsf{Lim}\,T \arrow[rr, "\mathsf{can}_F^T"'] \arrow[ur, "F\,\mathsf{proj}_i^T"'] & & \mathsf{Lim}\,FT \arrow[ul, "\mathsf{proj}_i^{FT}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F\,\mathsf{Lim}\,S \arrow[rr, "\mathsf{can}_F^S"] \arrow[dr, "F\,\mathsf{proj}_i^S"] \arrow[ddd, "Y_{\mathsf{Lim}\,S}"', "{[k]}"] & & \mathsf{Lim}\,FS \arrow[dl, "\mathsf{proj}_i^{FS}"'] \arrow[ddd, "{[k]}"', "\mathsf{Lim}\,YS"] \\ & FSi \arrow[d, "{[k]}"', "Y_{Si}"] & \\ & GSi & \\ G\,\mathsf{Lim}\,S \arrow[rr, "\mathsf{can}_G^S"'] \arrow[ur, "G\,\mathsf{proj}_i^S"'] & & \mathsf{Lim}\,GS \arrow[ul, "\mathsf{proj}_i^{GS}"] \end{tikzcd}$$

for every $i \in |J|$; the triangles are the definitions of the canonical arrows, the the trapezoids, from left to right are the F-image of the definition of $\mathsf{Lim}\,X$, the definition of $\mathsf{Lim}\,FX$, the n-naturality of Y at proj_i^S and the definition of $\mathsf{Lim}\,YS$.

To say the functor F is J-continuous is to assert that every canonical arrow can_F^S is an isomorphism, in which case I will write $\mathsf{nac}_F^S = [\mathsf{can}_F^S]^{-1}$. It is not enough to ask that $F\,\mathsf{Lim}\,S \approx \mathsf{Lim}\,FS$ for every S – we must ask that they be isomorphic in the right way. (The situation is akin to examples showing a Banach space can be isomorphic (in the wrong way) to its double-dual without being reflexive.)

For a set $\mathcal{J}$ of index categories, an n-category C is $\mathcal{J}$-complete when it is J-complete for every $J \in \mathcal{J}$. Likewise an n-functor F is $\mathcal{J}$-continuous when it is

J-continuous for every $J \in \mathcal{J}$. These are the 0- and 1-cells of the $(n+1)$-category $\mathsf{Cmp}_n^{\mathcal{J}}$ of $\mathcal{J}$-complete categories. The $(k+1)$-cells of $\mathsf{Cmp}_n^{\mathcal{J}}$ are arbitrary the n-natural k-transformations. (The n-naturality above shows that any k-transformation is automatically compatible with the limit process.) In the case $\mathcal{J} = |\,\mathsf{Cat}\,|$ we speak simply of complete n-categories and continuous n-functors.

The completeness of n-Cat will also be a prerequisite. Indexing by $i \in J$, let $\langle C_i \rangle_i$ be a n-Cat-valued functor. We have $|\,\mathsf{Lim}_i\, C_i| = \mathsf{Lim}_i\, |C_i|$ in Set, with elements all $|J|$-tuples $(a_i)_i = \langle a_i \rangle_{i \in |J|}$ with $a_i \in |C_i|$ and $C_m a_i = a_j$ for every $i \xrightarrow{m} j$ in J. The hom-$(n-1)$-categories of $\mathsf{Lim}_i\, C_i$ are $[\mathsf{Lim}_i\, C_i][(a_i)_i, (b_i)_i] = \mathsf{Lim}_i [C_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}$ where at $\mathsf{dim}\, i = 0$, $[C_i]_{a_i, b_i} = C_i(a_i, b_i)$. Indeed, given $i \xrightarrow{m} j \xrightarrow{n} j'$ in J,

$$\begin{aligned}
[C_{nm}]_{a_{\mathsf{dom}\, nm}, b_{\mathsf{dom}\, nm}} &= [C_n C_m]_{a_i, b_i} \\
&= [C_n]_{C_m a_i, C_m b_i} [C_m]_{a_i, b_i} \\
&= [C_n]_{a_j, b_j} [C_m]_{a_i, b_i} \\
&= [C_n]_{a_{\mathsf{dom}\, n}, b_{\mathsf{dom}\, n}} [C_m]_{a_{\mathsf{dom}\, m}, b_{\mathsf{dom}\, m}}.
\end{aligned}$$

Thus, an l-cell $(a_i)_i \xrightarrow[{[l]}]{(f_i)_i} (b_i)_i$ in $\mathsf{Lim}_i\, C_i$ is a $|J|$-tuple of l-cells $a_i \xrightarrow[{[l]}]{f_i} b_i$ in C_i for every $i \in |J|$. At such a cell the projections are defined as $\mathsf{proj}_i^{\langle C_i \rangle_i} (f_i)_i = f_i$.

For a fixed J-complete n-category C the J-completeness of every hom-n-category $C^A = n\text{-}\mathsf{Cat}(A, C)$ and the J-continuity of every translation $C^G = n\text{-}\mathsf{Cat}(G, C)$ for G an n-functor will also arise. In C^A, $\mathsf{Lim}_i\, F_i$ is given on cells a of A by the formula $[\mathsf{Lim}_i\, F_i]a = \mathsf{Lim}_i\, F_i a$, taking the latter limit in C. The projections are given by $\mathsf{proj}_i^{\langle F_i \rangle_i} a = \mathsf{proj}_i^{\langle F_i a \rangle}$. For a functor $A \xrightarrow{G} B$, for every cell a of A and every $i \in |J|$ we find that $\mathsf{proj}_i^{\langle F_i \rangle_i} Ga = \mathsf{proj}_i^{\langle F_i G a \rangle_i} = \mathsf{proj}_i^{\langle F_i G \rangle_i}$ so that the canonical arrow $\mathsf{can}_{\langle F_i \rangle_i}^{C^G}$, from $C^G\, \mathsf{Lim}_i\, F_i = (\mathsf{Lim}_i\, F_i) G$ to $\mathsf{Lim}_i\, C^G F_i = \mathsf{Lim}_i\, F_i G$, is the identity, proving C^G continuous.

6.2 Completeness and Continuity of Integrals

Under the right hypothesese, integral n-categories are J-complete, with limit structures as computable as the ingredients we use to build them.

Proposition 6.2.1 (Completeness Criterion)**.** *If* $\overline{A} \xrightarrow{U_\square} \mathsf{Cmp}_n^J$ *is an n-functor with* $A \in |\,\mathsf{Cmp}_n^J\,|$*, then* $\int^A U \in |\,\mathsf{Cmp}_n^J\,|$ *and indexing by* $i \in J$*, we may take*

- $\mathsf{Lim}_i(a_i, u_i) = \left(\mathsf{Lim}_i\, a_i, \mathsf{Lim}_i\, U_{\mathsf{proj}_{\mathsf{dom}\, i}^{\langle a_i \rangle_i}} u_i\right)$
- $\mathsf{proj}_i^{\langle (a_i, u_i)\rangle_i} = \left(\mathsf{proj}_i^{\langle a_i \rangle_i}, \mathsf{proj}_i^{\left\langle U_{\mathsf{proj}_{\mathsf{dom}\, i}^{\langle a_i \rangle_i}} u_i \right\rangle_i}\right)$ *for all* $i \in |J|$*, and*
- $\mathsf{lim}_i(f_i, p_i) = \left(\mathsf{lim}_i\, f_i, (\mathsf{lim}_i\, p_i)\, \mathsf{nac}_{U_{\mathsf{cod}_1 \mathsf{lim}_i f_i}}^{\left\langle U_{\mathsf{proj}_{\mathsf{dom}\, i}^{\langle a_i \rangle_i}} u_i \right\rangle_i}\right).$

Proof. We'll tame the notation as we go: it only looks messy because it has to stand alone in the statement of our proposition.

The first step is to write $U_f u$ as u^f. Since U is contravariant, this leads to pleasant formulas like $u^{fg} = (u^f)^g$ and $(g, q)(f, p) = (gf, q^{\mathsf{cod}_1 f} p)$.

Next, let $S = \langle (a_i, u_i)\rangle_i$, and $\pi_\square = \mathsf{proj}_\square^{\langle a_i \rangle_i}$. Shorten dom_0 to ∂.

The first thing to check is that the implied functors exist. Now, $\langle a_i \rangle_i = \mathsf{proj}_{A,U} \langle (a_i, u_i)\rangle_i$ is a functor to A by composition.

Let $a = \mathsf{Lim}_i\, a_i$, which exists since A is J-complete.

As for

$$Z = \left\langle U_{\mathsf{proj}_{\mathsf{dom}\, i}^{\langle a_i \rangle_i}} u_i \right\rangle_i = \langle u_i^{\pi_{\partial i}} \rangle_i,$$

consider cells $i \xrightarrow{m} j \xrightarrow{n} k$ of J.

Now, $u_i \in |U_{a_i}|$, so $Zi = u_i^{\pi_{\partial i}} = u_i^{\pi_i} \in |U_a|$ since $U_{a_i} \xrightarrow{U_{\pi_i}} U_a$. Also, $(a_i, u_i) \xrightarrow{(a_m, u_m)} (a_j, u_j)$ in $\int^A U$, so we have $u_i \xrightarrow{u_m} u_j^{a_m}$ in U_{a_i} so that $Zm = u_m^{\pi_{\partial m}} = u_m^{\pi_i}$ runs from Zi to $u_j^{a_m \pi_i} = u_j^{\pi j} = Zj$ so at least Z is a graph morphism into U_a.

Calculate

$$(a_{nm}, u_{nm}) = (a_n, u_n)(a_m, u_m) = (a_n a_m, u_n^{a_m} u_m)$$

by the functoriality of S and the composition of $\int^A U$ (since $\mathsf{dim}\, a_m = 1$). This gives that $u_{nm} = u_n^{a_m} u_m$. Hence,

$$\begin{aligned}
Z(nm) &= u_{nm}^{\pi_{\partial nm}} && \text{(def. of } Z\text{)} \\
&= u_{nm}^{\pi_i} && (\partial nm = i) \\
&= (u_n^{a_m} u_m)^{\pi_i} && \text{(above)} \\
&= u_n^{a_m \pi_i} u_m^{\pi_i} && (U \text{ is a functor}) \\
&= u_n^{\pi_j} u_m^{\pi_i} && (\pi \text{ is a cone}) \\
&= u_n^{\pi_{\partial n}} u_m^{\pi_{\partial m}} && \text{(vet } n \text{ and } m\text{)} \\
&= (Zn)(Zm) && \text{(def. of } Z\text{)}
\end{aligned}$$

Also,

$$(a_{1_i}, u_{1_i}) = 1_{(a_i, u_i)} = (1_{a_i}, 1_{u_i})$$

by the functoriality of S, whence $u_{1_i} = 1_{u_i}$ and

$$\begin{aligned}
Z1_i &= u_{1_i}^{\pi\partial 1_i} && \text{(def. of } Z\text{)} \\
&= u_{1_i}^{\pi_i} && \text{(vet } 1_i\text{)} \\
&= 1_{u_i}^{\pi_i} && \text{(above)} \\
&= 1_{u_i^{\pi_i}} && (U \text{ is a functor}) \\
&= 1_{u_i^{\pi\partial i}} && (\mathsf{dim}\, i = 0) \\
&= 1_{Zi} && \text{(def. of } Z\text{)}
\end{aligned}$$

completing the proof that Z is a functor into U_a.

Let $u = \mathsf{Lim}\, Z$, which exists since U_a is J-complete and write $\rho_\square = \mathsf{proj}_\square^Z$.

Now, $\langle(\pi_i, \rho_i)\rangle_i$ is a cone from (a, u) to S: in $\int^A U$,

$$\begin{tikzcd}
(a,u) \arrow[r, "(\pi_i,\rho_i)"] \arrow[d, "="'] & (a_i,u_i) \arrow[d, "(a_m,u_m)"] \\
(a,u) \arrow[r, "(\pi_j,\rho_j)"'] & (a_j,u_j)
\end{tikzcd}$$

for every $i \xrightarrow{m} j$ in J because

$$(a_m, u_m)(\pi_i, \rho_i) = (a_m\pi_i, u_m^{\pi_i}\rho_i) = (a_m\pi_i, u_m^{\pi\partial m}\rho_i) = (\pi_j, \rho_j)$$

both π and ρ being cones.

It remains only to see that $\langle(\pi_i, \rho_i)\rangle_i$ is the universal cone to S. Suppose $\langle(f_i, p_i)\rangle_i$ is an l-cone from (b, v) to $\langle(a_i, u_i)\rangle_i$. Let $i \xrightarrow{m} j$ be an arrow in J. In $\int_A U$ we have

$$\begin{tikzcd}
(b,v) \arrow[r, "(f_i,p_i)", "{[l]}"'] \arrow[d, "="'] & (a_i,u_i) \arrow[d, "(a_m,u_m)"] \\
(b,v) \arrow[r, "{[l]}", "(f_j,p_j)"'] & (a_j,u_j)
\end{tikzcd}$$

so that $(f_j, p_j) = (a_m, u_m)(f_i, p_i) = (a_m f_i, u_m^{\mathsf{cod}_1 f_i} p_i)$, whence $f_j = a_m f_i$ and $p_j = u_m^{\mathsf{cod}_1 f_i} p_i$. Thus, $\langle f_i \rangle_i$ is an l-cone from b to $\langle a_i \rangle_i$ and $\langle p_i \rangle_i$ is an l-cone from v to $\langle u_i^{\mathsf{cod}_1 f_{\partial i}} \rangle_i$. Thus we have $f = \mathsf{lim}_i\, f_i$ so that in A,

$$\begin{tikzcd}
b \arrow[r, "f", "{[l]}"'] \arrow[d, "="'] & a \arrow[d, "\pi_i"] \\
b \arrow[r, "{[l]}", "f_i"'] & a_i
\end{tikzcd}$$

and we have $p = (\mathsf{lim}_i\, p_i)\,\mathsf{nac}$ so that, in U_b,

$$\begin{tikzcd}
v \arrow[r, "p", "{[l]}"'] \arrow[rr, bend left, "\mathsf{lim}_i\, p_i", "{[l]}"'] \arrow[dr, "{[l]}", "p_i"'] & (\mathsf{Lim}_i\, u_i^{\pi\partial i})^{\mathsf{cod}_1 f} \arrow[d, "\rho_i^{\mathsf{cod}_1 f}"] & \mathsf{Lim}_i\, u_i^{\pi\partial i\, \mathsf{cod}_1 f} \arrow[l, "\mathsf{nac}"'] \arrow[dl, "\mathsf{proj}_i"] \\
& u_i^{\pi\partial i\, \mathsf{cod}_1 f} = u_i^{\mathsf{cod}_1 f\partial i} &
\end{tikzcd}$$

Therefore, $(\pi_i, \rho_i)(f, p) = (\pi_i f, \rho_i^{\mathsf{cod}_1 f} p) = (f_i, p_i)$, proving that the l-cone $\langle (f_i, p_i) \rangle_i$ factors through the cone $\langle (\pi_i, \rho_i) \rangle_i$.

Conversely, suppose $(\pi_i, \rho_i)(f, p) = (f_i, p_i)$ for all i. Then $\pi_i f = f_i$ and $\rho_i^{\mathsf{cod}_1 f} p = p_i$, forcing $f = \mathsf{lim}_i\, f_i$ and $p = (\mathsf{lim}_i\, p_i)\,\mathsf{nac}$. QED

Proposition 6.2.2 (Continuity Criterion). *Let* $(A, U) \xrightarrow{(F,P)} (B, V)$ *be an arrow of* cInt_n *with* $A \xrightarrow{F} B$ *in* Cmp_n^J*. Then* $\int^A U \xrightarrow{\int^F P} \int^B V$ *is in* Cmp_n^J*.*

Proof. Let $J \xrightarrow{\langle (a_i, u_i) \rangle_i} \int^A U$ be any functor, and write $(\pi_i, \rho_i) = \mathsf{proj}_i^{\langle (a_i, u_i) \rangle_i}$ for every $i \in |J|$. Taking $a = \mathsf{Lim}_i\, a_i = \mathsf{dom}_0\, \pi_i$ and indexing with $i \in J$, by the completeness criterion (6.2.1),

$$\begin{aligned}
\mathsf{can}^S_{\int^F P} &= \mathsf{lim}_i [\int^F P](\pi_i, \rho_i) \\
&= \mathsf{lim}_i (F\pi_i, P_a \rho_i) \\
&= (\mathsf{lim}_i\, F\pi_i, (\mathsf{lim}_i\, P_a \rho_i)\, \mathsf{nac}^{\left\langle V_{F\, \mathsf{proj}_{\mathsf{dom}\, i}^{\langle F a_i \rangle_i}} P_a \rho_i \right\rangle_i}_{V_{F\, \mathsf{cod}_1\, \mathsf{lim}_i\, F\pi_i}}).
\end{aligned}$$

Now, $\mathsf{lim}_i\, F\pi_i = \mathsf{can}_F^{\langle a_i\rangle}$ is invertible since F is J-continuous, as is $\mathsf{lim}_i\, P_a\rho_i = \mathsf{can}_{P_a}^{\langle U_{\pi_{\mathsf{dom}\, i}} u_i\rangle_i}$, by the J-continuity of $U_a \xrightarrow{P_a} V_{Fa}$ in Cmp_n^J.

Since both coordinates are invertible, the isomorphism criterion (3.11.1) applies to show that the canonical arrow is invertible. Every canonical arrow for $\int^F P$ takes this form, completing the proof. QED

6.3 Continuity of Integration

Proposition 6.3.1 (Continuity of Integration)**.** *Both integrals*

$$\mathsf{Int}_n \xrightarrow{\int_\Box} n\text{-}\mathsf{Cat}$$

and

$$\mathsf{cInt}_n \xrightarrow{\int^\Box} n\text{-}\mathsf{Cat}$$

are continuous $(n+1)$*-functors (in* Cmp'_{n+1}*).*

Proof. For every n-category A, $(n+1)\text{-}\mathsf{Cat}'(A, n\text{-}\mathsf{Cat})$ is complete, and for every n-functor F, $(n+1)\text{-}\mathsf{Cat}'(F, n\text{-}\mathsf{Cat})$ is continuous because $n\text{-}\mathsf{Cat}$ is complete. Therefore, by the completeness criterion (6.2.1) Int_n is complete, as is $\mathsf{cInt}_n \approx \mathsf{Int}_n^*$.

As usual, we will induct on n, proving the contravariant case from the covariant one. The following boilerplate is useful in both the base case and the induction.

To show that $\int_\Box$ is continuous, indexing with $i \in J$, let $J \xrightarrow{\langle (A_i, U^i_\Box)\rangle_i} \mathsf{Int}_n$ be any functor. By the completeness criterion (6.2.1) we have

$$\mathsf{Lim}_i(A_i, U^i) = (\mathsf{Lim}_i\, A_i, \mathsf{Lim}_i\, U^i\pi_{\mathsf{dom}\, i})$$

where $\pi_\Box = \mathsf{proj}^{\langle A_i\rangle_i}$. Write $\rho_\Box = \mathsf{proj}^{\langle U^i\pi_{\mathsf{dom}\, i}\rangle}$. We must confirm that

$$\mathsf{can} = \mathsf{can}_{\int_\Box}^{\langle (A_i, U^i)\rangle_i} = \mathsf{lim}_i \textstyle\int_\Box(\pi_i, \rho_i) = \mathsf{lim}_i \int_{\pi_i} \rho_i$$

is an isomorphism

$$\textstyle\int_{\square} \mathsf{Lim}_i (A_i, U^i) \xrightarrow{=} \int_{\mathsf{Lim}_i A_i} \mathsf{Lim}_i U^i \pi_{\mathsf{dom}\, i} \xrightarrow{\approx} \mathsf{Lim}_i \int_{A_i} U_i \,.$$

At $n = 0$ we calculate, for $((a_i)_i, (u_i)_i) \in \int_{\mathsf{Lim}_i A_i} \mathsf{Lim}_i U^i \pi_{\mathsf{dom}\, i}$ that

$$\begin{aligned}
(*)\, \mathsf{can}((a_i)_i, (u_i)i) &= [\mathsf{lim}_i \int_{\pi_i} \rho_i]((a_i)_i, (u_i)i) \\
&= ([\int_{\pi_i} \rho_i]((a_i)_i, (u_i)_i))_i \\
&= (\pi_i(a_i)_i, [\rho_i(a_i)_i](u_i)_i)_i \\
&= ((a_i, u_i))_i
\end{aligned}$$

It is immediate that can is injective. Let $((a_i, u_i))_i \in \mathsf{Lim}_i \int_{A_i} U_i$, and $i \xrightarrow{m} j$ in J. Now, $a_i \in A_i$, $u_i \in U_{a_i}$ and

$$(a_j, u_j) = [\textstyle\int_{A_m} U^m](a_i, u_i) = (A_m a_i, U^m_{a_i} u_i)$$

so that $a_j = A_m a_i$ and

$$u_j = U^m_{a_i} u_i = U^m_{\pi_i (a_i)_i} u_i = U^m_{\pi_{\mathsf{dom}\, m} (a_i)_i} u_i.$$

All told, we see $(a_i)_i \in \mathsf{Lim}_i A_i$ and $(u_i)_i \in \mathsf{Lim}_i U^i_{\pi_{\mathsf{dom}\, i}(a_i)_i} = [\mathsf{Lim}_i U^i \pi_{\mathsf{dom}\, i}](a_i)_i$ so that $((a_i)_i, (u_i)_i) \in \int_{\mathsf{Lim}_i A_i} \mathsf{Lim}_i U^i \pi_{\mathsf{dom}\, i}$, and (*) proves can is surjective, hence bijective.

Now let $n > 0$ and suppose the $(n-1)$-dimensional integrals are continuous. By

the case $n = 0$, $|\,\mathsf{can}\,| = \mathsf{lim}_i \int_{|\pi_i|} |\square|\rho_i$ is bijective. On a hom-$(n-1)$-category,

$$\begin{aligned}
&\left[\int_{\mathsf{Lim}_i\, A_i} \mathsf{Lim}_i\, U^i \pi_{\mathsf{dom}\, i}\right][((a_i)_i, (u_i)_i), ((b_i)_i, (v_i)_i)] \\
&= \int^{f \in [\mathsf{Lim}_i\, A_i][(a_i)_i, (b_i)_i]} [(\mathsf{Lim}_i\, U^i \pi_{\mathsf{dom}\, i})(b_i)_i][((\mathsf{Lim}_i\, U^i \pi_{\mathsf{dom}_i}) f)(u_i)_i, (v_i)_i] \\
&= \int^{f \in \mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}} [\mathsf{Lim}_i\, U^i_{b_{\mathsf{dom}\, i}}][(U^i_{\pi_i f} u_i)_i, (v_i)_i] \\
&= \int^{\mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}} \left\langle \mathsf{Lim}_i [U^i_{b_{\mathsf{dom}\, i}}]_{U^{\mathsf{dom}\, i}_{\pi_{\mathsf{dom}\, i} f} u_{\mathsf{dom}_i}, v_{\mathsf{dom}\, i}} \right\rangle^{f \in Lim_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}}
\end{aligned}$$

the action of can is

$$\begin{aligned}
&\left[\mathsf{lim}_i \int_{\pi_i} \rho_i\right]_{((a_i)_i, (u_i)_i), ((b_i)_i, (v_i)_i)} \\
&= \mathsf{lim}_i \left[\int_{\pi_i} \rho_i\right]_{((a_i)_i, (u_i)_i), ((b_i)_i, (v_i)_i)} \\
&= \mathsf{lim}_i \int^{[\pi_i]_{(a_i)_i, (b_i)_i}} \left\langle [\rho_i (b_i)_i]_{U^i_{\pi_i f} u_i, v_i} \right\rangle^{f \in \mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}}
\end{aligned}$$

which is the canonical $(n-1)$-functor for the contravariant integral at

$$\left(\mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}, \left\langle \mathsf{Lim}_i [U^i_{b_{\mathsf{dom}\, i}}]_{U^{\mathsf{dom}\, i}_{\pi_{\mathsf{dom}\, i} f} u_{\mathsf{dom}\, i}, v_{\mathsf{dom}\, i}} \right\rangle^{f \in \mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}} \right)$$

in cInt_{n-1} because

$$\begin{aligned}
[\pi_i]_{(a_i)_i, (b_i)_i} &= \left[\mathsf{proj}_i^{\langle A_i \rangle_i}\right]_{(a_i)_i, (b_i)_i} \\
&= \mathsf{proj}_i^{\langle [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}} \rangle_i}
\end{aligned}$$

and

$$
\begin{aligned}
&\Big\langle [\rho_i(b_i)_i]_{U^i_{\pi_i f} u_i, v_i} \Big\rangle_{f \in \mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}} \\
&= \Big\langle [\mathsf{proj}^{\langle U^i \pi_{\mathsf{dom}\, i} \rangle_i} (b_i)_i]_{U^i_{\pi_i f} u_i, v_i} \Big\rangle_f \\
&= \Big\langle [\mathsf{proj}_i^{\langle U^i_{b_{\mathsf{dom}\, i}} \rangle_i}]_{U^i_{\pi_i f} u_i, v_i} \Big\rangle_f \\
&= \Big\langle \mathsf{proj}_i^{\big\langle [U^i_{b_{\mathsf{dom}\, i}}]_{U^{\mathsf{dom}\, i}_{\pi_{\mathsf{dom}\, i} f} u_i, v_i} \big\rangle_i} \Big\rangle_f \\
&= \mathsf{proj}_i^{\Big\langle \big\langle [U^i_{b_{\mathsf{dom}\, i}}]_{U^{\mathsf{dom}\, i}_{\pi_{\mathsf{dom}\, i} f} u_i, v_i} \big\rangle^{f \in \mathsf{Lim}_i [A_i]_{a_{\mathsf{dom}\, i}, b_{\mathsf{dom}\, i}}} \Big\rangle_i} .
\end{aligned}
$$

Therefore, being bijective on 0-cells an an isomorphism on every hom-$(n-1)$-category, $\mathsf{can} = \mathsf{can}_{\int_\square}^{\langle (A_i, U^i) \rangle_i}$ is an isomorphism, as claimed.

In the contravariant case, $\int^\square$ is defined as the composite

$$\mathsf{cInt}_n \xrightarrow{\omega} \mathsf{Int}_n^* \xrightarrow{[\int_\square]^*} n\text{-}\mathsf{Cat}^* \xrightarrow{\overline{\square}^*} n\text{-}\mathsf{Cat}$$

of three continuous $(n+1)$-functors: ω and $\overline{\square}^*$ are continuous because they are isomorphisms, and $[\int_\square]^*$ is continuous because $\int_\square$ is continuous. QED

6.4 Sums of Integrals

Our integrals also respect coproducts, producing pleasing formulas when the coproducts are denoted by sums. Since coproducts are colimits, this seems worth mentioning here, but since the formulas are examples of exchanging the order of integration, it will be more efficient to treat them in chapter 7.

The key observation will be that when A is 0-dimensional,

$$\textstyle\sum_{a \in A} U_a = \int_{a \in A} U_a,$$

where each U_a is an n-category. One inclusion is immediate from proposition 5.1.3 (canonical embeddings of the n-categories U_a), the other from proposition 3.8.3 and induction, although these reasons may be more technical than the fact is. (A more direct consideration may be more convincing.)

6.5 Integrals of Composite Functors

Another miscellaneous limit result is the following.

Proposition 6.5.1 (Integrals of composites are pull-backs). *From the identity transformations, in* $(n+1)\text{-}\mathsf{Cat}'$,

$$\begin{tikzcd} A \arrow[r, "UF"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow 1_{UF}"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ B \arrow[r, "U_\square"'] & n\text{-}\mathsf{Cat} \end{tikzcd} \qquad \begin{tikzcd} \overline{A} \arrow[r, "V\overline{G}"] \arrow[d, "\overline{G}"'] \arrow[dr, phantom, "\Downarrow 1_{V\overline{G}}"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ \overline{B} \arrow[r, "V_\square"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

with both $A \xrightarrow{F} B$ *and* $A \xrightarrow{G} B$ *in* $n\text{-}\mathsf{Cat}$, *we obtain pull-back diagrams*

$$\begin{tikzcd} \int_A UF \arrow[r, "\int_F 1_{UF}"] \arrow[d, "\mathsf{proj}"'] & \int_B U \arrow[d, "\mathsf{proj}"] \\ A \arrow[r, "F"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \int^A V\overline{G} \arrow[r, "\int_G 1_{V\overline{G}}"] \arrow[d, "\mathsf{proj}"'] & \int^B V \arrow[d, "\mathsf{proj}"] \\ A \arrow[r, "G"'] & B \end{tikzcd}$$

in $n\text{-}\mathsf{Cat}$.

Proof. The squares commute by the naturality of the projections. Induct on n, working first on the covariant case. Supposing, in $n\text{-}\mathsf{Cat}$,

$$\begin{tikzcd} D \arrow[r, "Q"] \arrow[d, "P"'] & \int_B U \arrow[d, "\mathsf{proj}"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

we will show there is a unique n-functor R so that, in $n\text{-}\mathsf{Cat}$,

$$\begin{tikzcd}
D \arrow[dr, "R"] \arrow[drr, bend left, "Q"] \arrow[ddr, bend right, "P"'] & & \\
& \int_A UF \arrow[r, "\int_F 1_{UF}"] \arrow[d, "\mathsf{proj}"'] & \int_B U \arrow[d, "\mathsf{proj}"] \\
& A \arrow[r, "F"'] & B
\end{tikzcd}$$

In dimension $n = 0$, suppose R exists. Let $d \in D$ and write $Rd = (a_d, u_d)$. Now, $a_d = \mathsf{proj}\, Rd = Pd$ is determined by P, and u_d is determined since

$$(Fa_d, u_d) = [\textstyle\int_F 1_{UF}](a_d, u_d) = [\int_F 1_{UF}]Rd = Qd.$$

Conversely, taking a_d and u_d from P and Q as above does define a map, since $u_d \in U_{Fa_d}$ so that $Rd = (a_d, u_d) \in \int_A UF$. The contravariant case coincides.

Suppose $n > 0$ and inductively assume the proposition holds in dimension $n-1$. Thus, $|R|$ is uniquely determined by the case $n = 0$, if R exists. Again, write $|R|d = (a_d, u_d)$ for $d \in |D|$. Let $d, e \in |D|$, and consider the hom-$(n-1)$-categories

$$\mathcal{A} = [\textstyle\int_A UF][(a_d, u_d), (a_e, u_e)] = \int^{f \in A(a_d, a_e)} U_{Fa_e}(U_{Ff}u_d, u_e)$$

and

$$\mathcal{B} = [\textstyle\int_B U][(Fa_d, u_d), (Fa_e, u_e)] = \int^{g \in B(Fa_d, Fa_e)} U_{Fa_e}(U_g u_d, u_e).$$

Form the diagram of $(n-1)$-categories:

$$\begin{tikzcd}
D(d,e) \arrow[dr, "R_{d,e}"] \arrow[drr, bend left, "Q_{d,e}"] \arrow[ddr, bend right, "P_{d,e}"'] & & \\
& \mathcal{A} \arrow[r, "\int^{g = F_{a_d,a_e} f} 1_{U_{Fa_e}(U_{Ff}u_d, u_e)}"] \arrow[d, "\mathsf{proj}"'] & \mathcal{B} \arrow[d, "\mathsf{proj}"] \\
& A(a_d, a_e) \arrow[r, "F_{a_d,a_e}"'] & B(Fa_d, Fa_e)
\end{tikzcd}$$

The outside commutes by assumption and the inner square is a pull-back by induction, so there is a unique $(n-1)$-functor $R_{d,e}$ commuting as pictured. Thus, there is a unique

$(n-1)$-Cat-graph morphism R commuting as needed. It remains only to see that R is in fact an n-functor.

Take arbitrary k-cells of D,

$$c \xrightarrow[{[k]}]{\varphi} d \xrightarrow[{[k]}]{\gamma} e.$$

As before, write $R\gamma = (a_\gamma, u_\gamma)$ where $a_\gamma = P\gamma$ and $(Fa_\gamma, u_\gamma) = Q\gamma$. Now,

$$a_{\gamma\varphi} = P(\gamma\varphi) = (P\gamma)(P\varphi) = a_\gamma a_\varphi$$

and

$$\begin{aligned}(Fa_{\gamma\varphi}, u_{\gamma\varphi}) &= Q(\gamma\varphi)\\ &= (Q\gamma)(Q\varphi)\\ &= (Fa_\gamma, u_\gamma)(Fa_\varphi, u_\varphi)\\ &= ((Fa_\gamma)(Fa_\varphi), u_\gamma(U_{\mathsf{dom}_1 Fa_\gamma} u_\varphi))\end{aligned}$$

so that $u_{\gamma\varphi} = u_\gamma(U_{\mathsf{dom}_1 Fa_\gamma} u_\varphi)$, whence

$$\begin{aligned}R(\gamma\varphi) &= (a_{\gamma\varphi}, u_{\gamma\varphi})\\ &= (a_\gamma a_\varphi, u_\gamma(U_{\mathsf{dom}_1 Fa_\gamma} u_\varphi))\\ &= (a_\gamma, u_\gamma)(a_\varphi, u_\varphi) = (R\gamma)(R\varphi)\end{aligned}$$

Also,

$$a_{1_d} = P1_d = 1_{Pd} = 1_{a_d}$$

and

$$(Fa_{1_d}, u_{1_d}) = Q1_d = 1_{Qd} = 1_{(Fa_d, u_d)} = (1_{Fa_d}, 1_{u_d})$$

so that $u_{1_d} = 1_{u_d}$ and thus

$$R1_d = (a_{1_d}, u_{1_d}) = (1_{a_d}, 1_{u_d}) = 1_{(a_d,u_d)} = 1_{Rd}.$$

Thus, R is an n-functor.

Noting that n-Cat has pull-backs, we thus obtain a unique isomorphism between the canonical pull-back object and $\int_A UF$, proving the n-universal property defining a pull-back square in the covariant case.

As for the contravariant case, by the covariant case we have the pull-back in n-Cat^*,

$$\begin{tikzcd}
\int_{a\in\overline{A}^*} \overline{V_{Ga}}^* \arrow[r, "\int_{\overline{G}^*} 1_{\overline{VG}^*}"] \arrow[d, "\mathsf{proj}"'] & \int_{b\in\overline{B}^*} \overline{V_b}^* \arrow[d, "\mathsf{proj}"] \\
\overline{A}^* \arrow[r, "\overline{G}^*"'] & \overline{B}^*
\end{tikzcd}$$

and applying the continuous $(n+1)$-functor $\overline{\Box}^*$ we obtain the desired pull-back in n-Cat. QED

Chapter 7

Exchange of Integrals

Now I have the pleasure of proving certain results reminiscent of Fubini's famous theorem, that we can exchange the order of any two integrals, be they covariant, contravariant or both. Consequently, we may exchange the order of integration for any finite number of integrals.

7.1 Preliminaries

In the case of Fubini's theorem, one must first show that each single integral is a measurable function. For us, we must show that each single integral is an n-functor of the right variance.

Lemma 7.1.1 (Single integrals are n-functors)**.** *Let A and B lie in n-Cat.*

(i) *Let $A \times B \xrightarrow{U_\square} n\text{-}\mathsf{Cat}$ be an n-functor. Then both $\langle \int_{b\in B} U_{a,b}\rangle_{a\in A}$ and $\langle \int_{a\in A} U_{a,b}\rangle_{b\in B}$ are n-Cat-valued n-functors.*

(ii) *Let $\overline{A} \times \overline{B} \xrightarrow{V_\square} n\text{-}\mathsf{Cat}$ be an n-functor. Then both $\langle \int^{b\in B} V_{a,b}\rangle^{a\in A}$ and $\langle \int^{a\in A} V_{a,b}\rangle^{b\in B}$ are n-Cat-valued n-functors.*

(iii) *Let* $\overline{A} \times B \xrightarrow{W_\Box} n\text{-}\mathsf{Cat}$ *be an* n*-functor. Then both* $\langle \int_{b \in B} W_{a,b} \rangle^{a \in A}$ *and* $\langle \int^{a \in A} W_{a,b} \rangle_{b \in B}$ *are* $n\text{-}\mathsf{Cat}$*-valued* n*-functors.*

Proof. Ad (i), for the first functor, we have, in $n\text{-}\mathsf{Cat}'$,

$$\begin{tikzcd}
A \arrow[rr, "\langle \int_{b \in B} U_{a,b} \rangle_{a \in A}"] \arrow[d, "\mathsf{rad}\, U"'] & & n\text{-}\mathsf{Cat} \\
n\text{-}\mathsf{Cat}'(B, n\text{-}\mathsf{Cat}) \arrow[r, "="'] & (n+1)\text{-}\mathsf{Cat}'(B, n\text{-}\mathsf{Cat}) \arrow[r, "\mathsf{emb}_B"'] & \mathsf{Int}_n \arrow[u, "\int_\Box"']
\end{tikzcd}$$

Here, recall $\mathsf{rad}\, U$ is the right adjunct of U, from the definition of $n\text{-}\mathsf{Cat}$ as a closed category.

In other words, for $a \xrightarrow[{[k]}]{f} a'$ in A, we have defined

$$\langle \textstyle\int_{b \in B} U_{a,b} \rangle_{a \in A} f = \int_{b = b} U_{f,b} = \int_{1_B} \langle U_{f,b} \rangle_{b \in B}.$$

The case of $\langle \int_{a \in A} U_{a,b} \rangle_{b \in B}$ is similar, but using $\mathsf{rad}(U\beta)$ in place of $\mathsf{rad}\, U$ where

$$A \times B \xrightarrow[\approx]{\beta} B \times A$$

is the canonical isomorphism commuting with the projections.

As for (ii), consider the n-functor

$$\overline{A}^* \times \overline{B}^* \xrightarrow{V^*} n\text{-}\mathsf{Cat}^* \xrightarrow{\overline{\Box}^*} n\text{-}\mathsf{Cat}.$$

Applying (i) we get an n-functor $\langle \int_{b \in \overline{B}^*} \overline{V_{a,b}}^* \rangle_{a \in \overline{A}^*}$ from which we obtain the n-functor $\overline{\langle \int_{b \in \overline{B}^*} \overline{V_{a,b}}^* \rangle}^*_{a \in \overline{A}}$ which is $\langle \int^{b \in B} V_{a,b} \rangle^{a \in A}$. The other n-functor is similar.

For (iii), by (i) we have the n-functor

$$\langle \textstyle\int_{b \in B} W_{a,b} \rangle_{a \in \overline{A}} = \langle \int_{b \in B} W_{a,b} \rangle^{a \in A},$$

while by (ii) we have the n-functor

$$\langle \textstyle\int^{a \in A} W_{a,b} \rangle^{b \in \overline{B}} = \langle \int^{a \in A} W_{a,b} \rangle_{b \in B}.$$

QED

7.2 Homovariant Exchange of Integrals

In this section we show that the order of integration may be exchanged for two integrals of the same variance. This will be a consequence of the following:

Proposition 7.2.1 (Rectangular integration)**.** *For $A, B \in |n\text{-}\mathsf{Cat}|$, $(A \times B, U_\Box) \in |\mathsf{Int}_n|$ and $(A \times B, V_\Box) \in |\mathsf{cInt}_n|$, we have isomorphisms of n-categories*

$$\int_{(a,b)\in A\times B} U_{a,b} \xrightarrow[\approx]{\varphi} \int_{a\in A}\int_{b\in B} U_{a,b} \ \text{ and } \int^{(a,b)\in A\times B} V_{a,b} \xrightarrow[\approx]{\varphi} \int^{a\in A}\int^{b\in B} V_{a,b}\,,$$

both via $\varphi((a,b),u) = (a,(b,u))$ on cells.

Proof. For $n = 0$, the equivalence of the conditions

- $((a,b),u) \in \int_{(a,b)\in A\times B} U_{a,b}$
- $(a,b) \in A \times B$ and $u \in U_{a,b}$
- $a \in A$, $b \in B$ and $u \in U_{a,b}$
- $a \in A$ and $(b,u) \in \int_{b\in B} U_{a,b}$
- $(a,(b,u)) \in \int_{a\in A}\int_{b\in B} U_{a,b}$

shows that φ is a bijection. The same goes for contravariant integrals.

Consider the covariant case, with $n > 0$. Now, $|\varphi|$ is bijective by the case $n = 0$. On hom-$(n-1)$-categories, φ is

$$
\begin{array}{c}
[\int_{(a,b)\in A\times B} U_{a,b}][((a,b),u),((c,d),v)] \\
\Big\downarrow = \text{(def. of hom in } \textstyle\int) \\
\int^{(f,g)\in(A\times B)[(a,b),(c,d)]} U_{c,d}(U_{f,g}u,v) \\
\Big\downarrow = \text{(def. of hom in } A\times B) \\
\int^{(f,g)\in A(a,c)\times B(b,d)} U_{c,d}(U_{f,g}u,v) \\
\varphi \Big\downarrow \approx \\
\int^{f\in A(a,c)} \int^{g\in B(b,d)} U_{c,d}(U_{f,g}u,v) \\
\Big\downarrow = ((c,g)(f,b)=(f,g)) \\
\int^{f\in A(a,c)} \int^{g\in B(b,d)} U_{c,d}(U_{c,g}U_{f,b}u,v) \\
\Big\downarrow = \text{(def. of hom in } \textstyle\int_{b\in B} U_{c,b}) \\
\int^{f\in A(a,c)}[\int_{b\in B} U_{c,b}][(b,U_{f,b}u),(d,v)] \\
\Big\downarrow = \text{(action of } \textstyle\int_{b\in B} U_{f,b}) \\
\int^{f\in A(a,c)}[\int_{b\in B} U_{c,b}][(\int_{b\in B} U_{f,b})(b,u),(d,v)] \\
\Big\downarrow = \text{(def. of hom in } \textstyle\int) \\
[\int_{a\in A}\int_{b\in B} U_{a,b}][(a,(b,u)),(c,(d,v))]
\end{array}
$$

an isomorphism by induction. Thus, φ is a $(n-1)$-Cat-graph isomorphism. At a 0-cell $((a,b),u)$,

$$
\begin{aligned}
1_{\varphi((a,b),u)} &= 1_{(a,(b,u))} && \text{(def. of } \varphi) \\
&= (1_a, 1_{(b,u)}) && \text{(def. of identity)} \\
&= (1_a, (1_b, 1_u)) && \text{(def. of identity)} \\
&= \varphi((1_a, 1_b), 1_u) && \text{(def. of } \varphi) \\
&= \varphi(1_{(a,b)}, 1_u) && \text{(def. of identity)} \\
&= \varphi 1_{((a,b),u)} && \text{(def. of identity)}
\end{aligned}
$$

while for k-cells of $\int_{(a,b)\in A\times B} U_{a,b}$,

$$((a,b),u) \xrightarrow[{[k]}]{((f,g),p)} ((a',b'),u') \xrightarrow[{[k]}]{((f',g'),p')} ((a'',b''),u'')$$

we find

$$\begin{aligned}
\varphi[((f',g'),p')((f,g),p)] &= \varphi((f',g')(f,g),p'(U_{\mathsf{dom}_1 f',\mathsf{dom}_1 g'}p)) && \text{(def. of composition)}\\
&= \varphi((f'f,g'g),p'(U_{\mathsf{dom}_1 f',\mathsf{dom}_1 g'}p)) && \text{(def. of composition)}\\
&= (f'f,(g'g,p'(U_{\mathsf{dom}_1 f',\mathsf{dom}_1 g'}p))) && \text{(def. of } \varphi)\\
&= (f'f,(g'g,p'(U_{a'',\mathsf{dom}_1 g'}U_{\mathsf{dom}_1 f',b'}p))) && (U \text{ is a functor})\\
&= (f'f,(g',p')(g,U_{\mathsf{dom}_1 f',b'}p)) && \text{(def. of composition)}\\
&= (f'f,(g',p')\left[\left(\int_{b\in B} U_{\mathsf{dom}_1 f',b}\right)(g,p)\right]) && \text{(preliminaries)}\\
&= (f',(g',p'))(f,(g,p)) && \text{(def. of composition)}\\
&= [\varphi((f',g'),p')][\varphi((f,g),p)] && \text{(def. of } \varphi)
\end{aligned}$$

which shows φ to be an isomorphism of n-categories.

In the contravariant case, φ is given as

$$
\begin{tikzcd}
\int^{(a,b)\in A\times B} V_{a,b} \arrow[d, "= \text{(def. of } \int^{\Box}\text{)}"] \\
\overline{\int_{(a,b)\in \overline{A}^* \times \overline{B}^*} \overline{V_{a,b}}^*}^* \arrow[d, "\approx", "\overline{\varphi}^*"'] \\
\overline{\int_{a\in\overline{A}^*} \int_{b\in\overline{B}^*} \overline{V_{a,b}}^*}^* \arrow[d, "= (\overline{\Box}^* \text{ is self-inverse})"] \\
\overline{\int_{a\in\overline{A}^*} \overline{\overline{\int_{b\in\overline{B}^*} \overline{V_{a,b}}^*}^*}^*}^* \arrow[d, "= \text{(def. of } \int^{a\in A}\text{)}"] \\
\int^{a\in A} \overline{\int_{b\in\overline{B}^*} \overline{V_{a,b}}^*}^* \arrow[d, "= \text{(def. of } \int^{b\in B}\text{)}"] \\
\int^{a\in A} \int^{b\in B} V_{a,b}
\end{tikzcd}
$$

QED

Corollary 7.2.2 (Homovariant exchange of integrals)**.** *For $A, B \in |n\text{-}\mathsf{Cat}\,|$, $(A \times B, U_{\Box}) \in |\,\mathsf{Int}_n\,|$ and $(A \times B, V_{\Box}) \in |\,\mathsf{cInt}_n\,|$, we have isomorphisms of n-categories*

$$\int_{a\in A} \int_{b\in B} U_{a,b} \xrightarrow{\quad\approx\quad} \int_{b\in B} \int_{a\in A} U_{a,b}$$

and

$$\int^{a\in A} \int^{b\in B} V_{a,b} \xrightarrow{\quad\approx\quad} \int^{b\in B} \int^{a\in A} V_{a,b}\,,$$

in both cases given by $(a,(b,u)) \mapsto (b,(a,u))$.

Proof. By rectangular integration and the functoriality of the integral on the canonical $A \times B \approx B \times A$,

$$\begin{tikzcd}
\int_{a\in A}\int_{b\in B} U_{a,b} \arrow[d, "\varphi^{-1}"', "\approx"] \\
\int_{(a,b)\in A\times B} U_{a,b} \arrow[d, "\approx"] \\
\int_{(b,a)\in B\times A} U_{a,b} \arrow[d, "\varphi"', "\approx"] \\
\int_{b\in B}\int_{a\in A} U_{a,b}
\end{tikzcd}$$

The contravariant case is the same, *mutatis mutandis* . QED

7.3 Heterovariant Exchange of Integrals

We now show the order of integration can be reversed for integrals of differing variances.

Proposition 7.3.1 (Heterovariant exchange of integrals)**.** *For an n-functor*

$$\overline{A} \times B \xrightarrow{U_\Box^\Box} n\text{-}\mathsf{Cat}$$

with $A, B \in |n\text{-}\mathsf{Cat}\,|$, writing $U_b^a = U(a, b)$, we have an isomorphism of n-categories

$$\int^{a\in A}\int_{b\in B} U_b^a \xrightarrow[\approx]{\varphi} \int_{b\in B}\int^{a\in A} U_b^a\,,$$

given on cells by $\varphi(a,(b,u)) = (b,(a,u))$.

Proof. In the case $n = 0$ we are faced with a homovariant exchange of integrals, which we have already proved valid.

For $n > 0$, we know $|\varphi|$ is bijective by the case $n = 0$. On hom-$(n-1)$-categories we find inductively that φ acts as

$$\begin{array}{c}
[\int^{a\in A}\int_{b\in B}U^a_b][(a,(b,u)),(c,(d,v))] \\
\Big\downarrow = \text{(def. of hom in } \int^{a\in A}) \\
\int_{f\in A(a,c)}[\int_{b\in B}U^a_b][(b,u),(\int_{b\in B}U^f_b)(d,v)] \\
\Big\downarrow = \text{(action of } \int_{b\in B}U^f_b) \\
\int_{f\in A(a,c)}[\int_{b\in B}U^a_b][(b,u),(d,U^f_d v)] \\
\Big\downarrow = \text{(def. of hom in } \int_{b\in B}) \\
\int_{f\in A(a,c)}\int^{g\in B(b,d)}U^a_d(U^a_g u, U^f_d v) \\
{\scriptstyle \varphi^{-1}}\Big\downarrow \approx \\
\int^{g\in B(b,d)}\int_{f\in A(a,c)}U^a_d(U^a_g u, U^f_d v) \\
\Big\downarrow = \text{(def. of hom in } \int^{a\in A}) \\
\int^{g\in B(b,d)}[\int^{a\in A}U^a_d][(a,U^a_g u),(c,v)] \\
\Big\downarrow = \text{(action of } \int^{a\in A}U^a_g) \\
\int^{g\in B(b,d)}[\int^{a\in A}U^a_d][(\int^{a\in A}U^a_g)(a,u),(c,v)] \\
\Big\downarrow = \text{(def. of hom in } \int_{b\in B}) \\
[\int_{b\in B}\int^{a\in A}U^a_b][(b,(a,u)),(d,(c,v))]
\end{array}$$

and so φ is a $(n-1)$-Cat-graph isomorphism. At a 0-cell $(a,(b,u))$ we see

$$\begin{aligned}
\varphi 1_{(a,(b,u))} &= \varphi(1_a, 1_{(b,u)}) && \text{(def. of identity)} \\
&= \varphi(1_a,(1_b,1_u)) && \text{(def. of identity)} \\
&= (1_b,(1_a,1_u)) && \text{(def. of } \varphi) \\
&= (1_b, 1_{(a,u)}) && \text{(def. of identity)} \\
&= 1_{(b,(a,u))} && \text{(def. of identity)} \\
&= 1_{\varphi(a,(b,u))} && \text{(def. of } \varphi)
\end{aligned}$$

while for k-cells of $\int^{a\in A}\int_{b\in B}U^a_b$,

$$(a,(b,u)) \xrightarrow[[k]]{(f,(g,p))} (a',(b',u')) \xrightarrow[[k]]{(f',(g',p'))} (a'',(b'',u'')),$$

we have

$$\begin{aligned}
\varphi[(f',(g',p'))(f,(g,p))] &= \varphi(f'f, \left[\left(\int_{b\in B} U_b^{\mathsf{cod}_1 f}\right)(g',p')\right](g,p)) && \text{(compose outside)}\\
&= \varphi(f'f,(g',U_{b''}^{\mathsf{cod}_1 f}p')(g,p)) && \text{(action of integral)}\\
&= \varphi(f'f,(g'g,(U_{b''}^{\mathsf{cod}_1 f}p')(U^a_{\mathsf{dom}_1 g'}p))) && \text{(compose inside)}\\
&= (g'g,(f'f,(U_{b''}^{\mathsf{cod}_1 f}p')(U^a_{\mathsf{dom}_1 g'}p))) && \text{(def. of } \varphi)\\
&= (g'g,(f',p')(f,U^a_{\mathsf{dom}_1 g'}p)) && \text{(factor inside)}\\
&= (g'g,(f',p')\left[\left(\int^{a\in A} U^a_{\mathsf{dom}_1 g'}\right)(f,p)\right]) && \text{(action of integral)}\\
&= (g',(f',p'))(g,(f,p)) && \text{(factor outside)}\\
&= [\varphi(f',(g',p'))][\varphi(f,(g,p))] && \text{(def. of } \varphi)
\end{aligned}$$

showing that φ is an n-functor, hence an isomorphism of n-categories. QED

7.4 An Alternate Development

An alert reader will have undoubtedly noticed that we could have made do with just the heterovariant mixed integral from the beginning. In detail, if we defined a double integral

$$\int\int_{b\in B}^{a\in A} U_b^a = \int^{a\in A}\int_{b\in B} U_b^a$$

recoordinatized with cells (a,b,u) replacing cells $(a,(b,u))$ we would have hom-$(n-1)$-categories

$$\left[\int\int_{b\in B}^{a\in A} U_b^a\right][(a,b,u),(c,d,v)] = \int\int_{f\in A(a,c)}^{g\in B(b,d)} U_d^a(U_g^a u, U_d^f v)$$

which are also double-integrals, obviating the need to weave the covariant and contravariant integrals side by side, since they could be recovered by taking A or B as the terminal n-category $\mathbb{1}$. These double-integral n-categories correspond exactly to bifibred n-categories without the need to reverse cells in every other dimension.

On the other hand, the continuity of the contravariant integral translates through duality to something awkward for the covariant integral, and these double-integrals are not as common in practice as their single counterparts. Moreover, the sensory overload resulting from this development (where the contravariance details alone are difficult to notate) led me to separate the integrals for clarity.

I wish to state, however, a few foundational examples, using these double (or if you prefer, heterovariant) integrals, that I find quite intriguing. In order to do this, let me write

$$\int\int_{b\in B}^{a\in A} U_b^a \xrightarrow{\mathsf{proj}} A \times B$$

for the n-functor with components isomorphic to the projections from the iterated integrals in both orders.

First, let me mention two propositions which we would encounter along this alternate development.

Proposition 7.4.1 (Biintegration is $(n+1)$-functorial)**.** *Let*

$$\mathsf{bInt} = \int^{(A,B)\in \widetilde{n\text{-}\mathsf{Cat}} \times n\text{-}\mathsf{Cat}} (n+1)\text{-}\mathsf{Cat}'(\overline{A} \times B, n\text{-}\mathsf{Cat})$$

be the (single integral!) $(n+1)$-category of n-dimensional "biintegrands," where here $\widetilde{n\text{-}\mathsf{Cat}}$ is $n\text{-}\mathsf{Cat}$ with only the 2-cells reversed. Then the double integral $\int\int_{\square}^{\square}$ is an $(n+1)$-functor from bInt to $n\text{-}\mathsf{Cat}$.

Proposition 7.4.2 (Biintegral k-cells are biintegrals)**.** *If a k-cell Φ is biintegral over k-cells F and G in the sense that*

$$\begin{tikzcd} \int\int_{b\in B}^{a\in A} U_b^a \arrow[r, "\Phi", "{[k]}"'] \arrow[d, "\mathsf{proj}"'] & \int\int_{d\in D}^{c\in C} V_d^c \arrow[d, "\mathsf{proj}"'] \\ A\times B \arrow[r, "F\times G", "{[k]}"'] & C\times D \end{tikzcd}$$

then there is an n-natural k-transformation

$$\begin{tikzcd} \overline{A}\times B \arrow[r, "U_\square^\square"] \arrow[d, "F\times G"', "{[k]}"] \arrow[dr, phantom, "{[k+1]}\Downarrow P_\square"] & n\text{-}\mathsf{Cat} \arrow[d, "="] \\ \overline{C}\times D \arrow[r, "V_\square^\square"'] & n\text{-}\mathsf{Cat} \end{tikzcd}$$

so that $(A,B,U)\xrightarrow[{[k]}]{(F,G,P)}(C,D,V)$ *lies in* bInt *and* $\Phi = \int\int_G^F P$.

Example 7.4.3 (n-Natural Transformations)**.**

The n-natural transformation $\begin{tikzcd} A \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow X"] & B \end{tikzcd}$ in n-Cat is the same as a diagram

$$\begin{tikzcd} A \arrow[rr, "X'"] \arrow[dr, "\Delta"'] & & \int\int_{a\in A}^{b\in A} |C(Sa,Tb)| \arrow[dl, "\mathsf{proj}"] \\ & A\times A & \end{tikzcd}$$

where $\Delta = 1_A \dot{\times} 1_A$ is the diagonal n-functor (acting on cells by $\Delta x = (x,x)$); the correspondence is given as

$$X = \langle Sa \xrightarrow{x'a} Ta\rangle_{a\in A}$$

where $X'a = (a,a,x'a)$ for $a\in|A|$. Indeed, for k-cells f in A ($k>0$), $X'f = (f,f,1)$ asserts the n-naturality of X at f.

Example 7.4.4 (Lawvere's Higher Adjunction Criterion)**.**

For $B\xrightarrow{F}A\xrightarrow{U}B$ in n-Cat, a diagram

$$\begin{tikzcd}
\int \int_{a\in A}^{b\in B} |A(Fb,a)| \arrow[rr, "\Phi'", "\approx"'] \arrow[dr, "\mathsf{proj}"'] & & \int \int_{a\in A}^{b\in B} |B(b,Ua)| \arrow[dl, "\mathsf{proj}"] \\
& A \times B &
\end{tikzcd}$$

is the same as an n-adjunction

$$A(Fb,a) \xrightarrow[\approx]{\Phi} B(b,Ua)$$

(F left n-adjoint to U).

In fact, this is a direct generalization of exercise IV. 1. 2 in [9], which is easily proved by adding $[k]$'s to proposition III. 2. 1 (the Yoneda lemma) and to either theorem IV. 2. 2(ii) or (iv) (the left and right adjoint criteria). Alternately, one can apply proposition 7.4.2.

Chapter 8

Further Directions

The pressing issue of how to define *weak* n-category theory is the topic of the book [8], which is lengthy mostly due to the large number of competing definitions and their complexity. The topic is further confused by the fact that category theory defines the notion of equivalence of definitions. A typical result takes the form "definition A is equivalent (under definition A's notion of equivalence) to definition B, when interpretted within definition A in a particular way." Of course, the number of pairings of definitions already seems hard to manage, so if any of them could be excluded, even temporarily, that would be a boon to the field. The integral, involving only one definition of weak n-category theory at a time, can serve as a useful litmus test to focus effort on those definitions which can support it.

(Those who have never been exposed to the complexities of coherence theory, which is the hallmark of weak categorical thought, may wish to consider an early such problem: Faced with a bifunctor which is associative up to a nonidentity natural isomorphism, what conditions do we need to check to guarantee we can ignore the lack of associativity on the nose? A good concrete example is the tensor product of modules over a commutative ring: how do we prove no trouble will ever come from

failing to parenthesize them? Or perhaps more to the point, how do we prove only wasted effort will be involved in always completely parenthesizing them?)

It is easy to adapt the 1-dimensional case to weak monoidal categories. Coherence conditions for the higher-dimensional transformations in the higher categories of weakly monoidal strict n-categories become immediately apparent upon trying to proceed to larger n, but the diagrams involved are large (and infinite in number, as n is unbounded). Untangling these coherence conditions could exclude (even if only temporarily) some of the competition for the most general definition.

A more obvious first step is to weaken the n-functor integrated while retaining the strictness of the resulting integral n-category. (For this, the integrand will probably have to be strict-n-category-valued.) F. Borceux ([2]) offers a good look at this program in dimension 1, as originated by Grothendieck whose original application required this level of generality. Unfortunately, the equations involved will be an order of magnitude more complicated than those explored in this thesis.

Together, these last two problems should be very good preparation for the most general problem of completely weakening the higher integral. And, if we are lucky enough that something as good as the integral can survive the passage into a weak theory, this should help focus us on that definition of weak higher category theory, even if the most general definition eventually selected is not so fortunate.

Bibliography

[1] M. Brittenham, *personal communication*

[2] F. Borceux, Handbook of Categorical Algebra 2, *Encyclopedia of Mathematics and its Applications, no. 51.*

[3] S. W. Dyer, The Strict Higher Grothendieck Integral, *Ph.D. Thesis.*

[4] A. Grothendieck, Categories fibrées et descente, *Seminaire de geometrie algebrique de l'Institut des Hautes Etudes Scientifiques,* Paris, 1961.

[5] C. Hermida, Some Properties of Fib as a Fibred 2-Category, *Journal of Pure and Applied Algebra, 134* (1), 83-109, 1999.

[6] G. M. Kelly, Basic Concepts of Enriched Category Theory, *LMS Lecture Note Series, 64.*

[7] T. Kuhn, The Structure of Scientific Revolutions, *University of Chicago Press*, 1970.

[8] T. Leinster, Higher Operads, Higher Categories, *LMS Lecture Notes Series, 298.*

[9] S. Mac Lane, Categories for the Working Mathematician, *Springer-Verlag GTM, 5.*

[10] K. S. S. Nambooripad, Theory of cross-connections, Centre for Mathematical Sciences, *Trivandrum, Kerala, India,* Publication Number 28, (1994).

[11] R. Vale, On the Opposite of the Category of Rings, arXive:0806.1476